(2) Complex Lie groups.

If G is a complex manifold endowed with holomorphic maps

$$G \times G \xrightarrow{m} G \quad \text{(multiplication)}$$
$$G \xrightarrow{i} G \quad \text{(inverse)}$$

satisfying usual axioms for a group, it is a <u>complex Lie group</u>.

subexamples: $(\mathbb{C}^n, +)$, $((\mathbb{C}^*)^n, \cdot)$, $GL_n(\mathbb{C})$, <u>not</u> $U(n)$.

(3) <u>Quotients</u> If X is a complex manifold, G is a complex Lie group, and $G \times X \xrightarrow{a} X$ is a holomorphic action map, we can <u>hope</u> to form the quotient X/G. As with C^∞-manifolds, we impose conditions to ensure it works:

Def The action $a: G \times X \to X$ is **free** if $\forall x \in X$, $g \cdot x = x \Rightarrow g = 1$.

The action is **proper** if the map

$$G \times X \xrightarrow{(a, pr_2)} X \times X$$

$$(g, x) \mapsto (g \cdot x, x) \quad \text{is}$$

a proper map (i.e. inverse images of compact sets are compact sets).

Prop If $G \times X \xrightarrow{a} X$ is a proper, free action of a complex Lie group on a complex manifold, then X/G is naturally a complex manifold in such a way that the projection

$$X \xrightarrow{\pi} X/G \quad \text{is holomorphic.}$$

Ex $\mathbb{C}^*$ acts properly on $\mathbb{C}^{n+1} \setminus \{0\}$ (and freely, of course!).

Non-example: $\mathbb{C}^*$ does not act freely on $\mathbb{C}$ (by rescaling). Quotient would be non-Hausdorff! [TN: explain]

Consider the action of the (unipotent) Lie group

$$G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{C} \right\} \text{ on } \mathbb{C}^2:$$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ay \\ y \end{pmatrix}.$$

Then $G \cong \mathbb{C}$ as complex manifolds (and even, $G \cong (\mathbb{C}, +)$ as Lie groups).

The map $G \times \mathbb{C}^2 \xrightarrow{(\alpha, p_2)} \mathbb{C}^2 \times \mathbb{C}^2$

is $\mathbb{C}^3 \longrightarrow \mathbb{C}^4$

$$(a, x, y) \longmapsto (x+ay, y, x, y).$$

This is <u>not</u> proper:

$$(\alpha, p_2)^{-1}(0,0,0,0) = \mathbb{C} \times \{0\} \times \{0\}.$$

<u>Complex tori</u>

Let $\{v_1, \ldots, v_{2n}\} \subset \mathbb{C}^n$ be a basis as a <u>real</u> vector space. Let

$$\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_{2n} \subset \mathbb{C}^n.$$

Λ is a subgroup of $(\mathbb{C}^n, +)$.

<u>Claim</u> Λ acts freely and properly on $\mathbb{C}^n$.

<u>Pf sketch</u> "~~F~~freely" is clear.

"Properly": use

<u>Lemma</u> For all $\varepsilon > 0$,

$$\Lambda \cap B_\varepsilon(0) = \{v \in \Lambda \mid |v| < \varepsilon\}$$ is finite.

I'll explain pf of lemma when $n=1$ ...
just gets more complicated with bookkeeping
for larger n.

let $v_2^\perp$ be orthogonal proj'n of
v_2 <u>away</u> from $\text{span}(v_1)$, so

$$v_2 = v_2^\perp + \lambda v_1 \quad \text{where } \lambda \in \mathbb{R},$$

and $\langle v_2^\perp, v_1 \rangle = 0$. Then

$$|av_1 + bv_2| = |a+b\lambda||v_1| + |b||v_2|,$$

for all $a, b \in \mathbb{Z}$. Thus there are
only finitely many b for which $\exists\, a$ s.t.
$|av_1 + bv_2| < \varepsilon$, and for each fixed
b there can be only finitely many
such a.

It follows from the lemma that

Λ is a <u>discrete</u> subgroup of $\mathbb{C}^n$ (in induced topology).

Now consider the image of

$$\Lambda \times \mathbb{C}^n \xrightarrow{(a, p_2)} \mathbb{C}^n \times \mathbb{C}^n.$$

Image is

$$\mathrm{Im}(a, p_2) = \coprod_{\gamma \in \Lambda} \left((\gamma, 0) + \Delta \right)$$

where $\Delta = \{ (v, v) \mid v \in \mathbb{C}^n \}$.

Now $\Delta \subset \mathbb{C}^n \times \mathbb{C}^n$ is closed, and since Λ is discrete, one sees that $\mathrm{Im}(a, p_2)$ is closed [since any two of the subsets $(\gamma_1, 0) + \Delta$, $(\gamma_2, 0) + \Delta$ lie a nonzero distance apart, a limit point of a sequence in $\mathrm{Im}(a, p_2)$ must be a limit point of a sequence in a <u>single</u> $(\gamma, 0) + \Delta$, hence lies in $(\gamma, 0) + \Delta$].

So the action of Λ on $\mathbb{C}^n$ is proper! $\square$

<u>Rmk</u> $\mathbb{C}^n/\Lambda$ is a complex manifold which may or may not be a projective algebraic manifold!

<u>Grassmannians</u> Let

$$Gr_k(\mathbb{C}^n) = \{ \underset{\text{linear subspace}}{W \subset \mathbb{C}^n} \mid \dim W = k \}.$$

This is the Grassmannian of k-planes in $\mathbb{C}^n$.

The group $GL_n\mathbb{C}$ acts transitively on the set $Gr_k(\mathbb{C}^n)$. The stabilizer of $\mathrm{span}(e_1, \dots, e_k) \subset \mathbb{C}^n$ is the subgroup

$$P_k = \left\{ \begin{array}{cc} \overbrace{\quad}^{k} & \\ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) & \}\, n-k \end{array} \right\} \subset GL_n\mathbb{C}.$$

This is a closed subgroup of a complex Lie group, and it follows that $GL_n\mathbb{C}/P_k$ is naturally a complex manifold.

[Pf that action is proper: consider

$$P_k \times G \xrightarrow{(a, p_2)} G \times G.$$

$$(\gamma, g) \longmapsto (\gamma g, g).$$

This is an injection. Its image is the inverse image, under the map

$$G \times G \longrightarrow G$$
$$(h, g) \longmapsto hg^{-1},$$

of the subset $P_k \subseteq G$. If $P_k \subseteq G$ is a closed subgroup, then $\mathrm{Im}(a, p_2)$ is closed, so the action is proper.]

Get charts on $Gr_k(\mathbb{C}^n)$ as follows: given $W \subset \mathbb{C}^n$, choose complementary $V \subset \mathbb{C}^n$ so $\mathbb{C}^n = W \oplus V$.

Get a map

$$\mathrm{Hom}_{\mathbb{C}}(W, V) \longrightarrow Gr_k(\mathbb{C}^n) \text{ by}$$

$$\varphi \longmapsto \{(w, \varphi(w)) \mid w \in W\} \subset \mathbb{C}^n.$$

This is an isomorphism onto an open subset of $Gr_k(\mathbb{C}^n)$.