A cool application of Hard Lefschetz

Recall A spectral sequence is a sequence $\{E_r, d_r\}_{r\geq 0}$ of bigraded groups

$$E_r = \bigoplus_{p,q} E_r^{p,q}$$

with differentials

$$d_r : E_r^{pq} \longrightarrow E_r^{p+r,q-r+1},$$

$$d_r^2 = 0$$

with isomorphisms $H^\cdot(E_r, d_r) = E_{r+1}$.

If $E_r = E_{r+1} = \dots$ for $r >> 0$, we say $E.$ converges and write E_∞ for this limiting case. The spectral sequence degenerates at E_r if $d_s = 0$ for $s \geq r$.

<u>Typical Example</u> A complex w/ descending filtration $(F^p C^\bullet, d)$ (so $d(F^p(C^n)) \subset F^p(C^{n+1})$)

gives

$$Z_r^{p,q} = \ker\left(d : F^p C^{p+q} \longrightarrow \frac{C^{p+q+1}}{F^{p+r} C^{p+q+1}}\right)$$

$$B_r^{p,q} = \operatorname{Im}\left(d : F^{p-r} C^{p+q-1} \longrightarrow C^{p+q}\right) \cap F^p C^{p+q}.$$

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q} + Z_r^{p+1,q-1}}.$$

Also, very generally,

Grothendieck spectral sequence

$$E_2^{p,q} = R^p G \circ R^q F \Longrightarrow R^{p+q}(G \circ F).$$

Sub-example Leray spectral sequence.

$X \xrightarrow{f} Y$ a map, $\mathcal{F}$ a sheaf of ab. gps on X. Direct image

$$(f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}V).$$

This gives functor

$$f_* : \mathrm{AbShv}(X) \longrightarrow \mathrm{AbShv}(Y).$$

It has right derived functors

$$\mathbb{R}^i f_* : \mathrm{Ab\,Shv}(X) \longrightarrow \mathrm{Ab\,Shv}(Y).$$

Main example $X \xrightarrow{f} Y$ a proper submersion of manifolds. Then $\mathbb{R}^i f_* \mathbb{Z}$, $\mathbb{R}^i f_* \mathbb{R}$, $\mathbb{R}^i f_* \mathbb{C}$ are sheaves whose stalk over $y \in Y$ is

$$(\mathbb{R}^i f_* A)_y \cong H^i(f^{-1}(y), A).$$

e.g. $Y = pt$, then

$$R^i f_* A \cong H^i(X, A).$$

<u>Leray s.s:</u>

$$E_2^{pq} = H^p(Y, R^q f_* A) \Rightarrow H^{p+q}(X, A).$$

<u>Thm</u> [Deligne] If $f: X \to Y$ is a proper submersion of Kähler manifolds, then Leray s.s. for constant sheaf $\mathbb{C}$ degenerates at E_2.

[If f is smooth projective morphism of smooth $\mathbb{C}$-varieties, then also for $\mathbb{Q}$.]

<u>Cor</u>

$$h^i(X, \mathbb{C}) = \sum_{p+q=i} h^p(Y, R^q f_* \mathbb{C}).$$

Pf sketch   Main point: have Kähler form ω on X. Check that

(i) wedging w/ ω defines

$$E_r^{p,q} \xrightarrow{L} E_r^{p,q+2} \quad \forall p,q,r.$$

(ii) $[L, d_r] = 0 \quad \forall r.$

Requires writing Leray using forms.

Define subsheaf

$$P^q \subset R^q f_* \mathbb{C} \text{ of primitive classes.}$$

Enough to check that d_r vanishes on $H^p(Y, P^{n-k}) \ \forall\ r \geq 2.$

Write $R^q = R^q f_* \mathbb{C}$. Consider

$$d_r : H^p(Y, P^{n-k}) \to H^{p-r}(Y, R^{n-k-r+1}).$$

Well, consider

$$\begin{tikzcd}
H^p(Y, P^{n-k}) \arrow[r, "d_r"] \arrow[d, "L^{k+r-1}"] & H^{p+r}(Y, R^{n-k-r+1}) \arrow[d, "L^{k+r-1}", "\wr"'] \\
H^p(Y, R^{n+k+2r-2}) \arrow[r, "d_r"] & H^{p+r}(Y, R^{n+k+r-1}).
\end{tikzcd}$$

Left-hand vertical arrow is zero for $r \geq 2$ because of Hard Lefschetz for primitive forms.

Right-hand vertical arrow is an isomorphism by Hard Lefschetz.

So, must have top $d_r = 0$ for $r \geq 2$. $\square$

Given family $X \xrightarrow{f} Y$ as above, and $y \in Y$, get <u>monodromy action</u> of $\pi_1(Y, y)$ on $H^\bullet(X_y, \mathbb{C})$ $(X_y = f^{-1}(y))$.

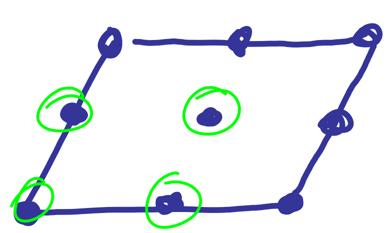
Cor If the monodromy action of $\pi_1(Y, y)$ on $H^\bullet(X_y, \mathbb{C})$ is trivial then
$$h^i(X, \mathbb{C}) = \sum_{p+q=i} h^p(Y, \mathbb{C}) h^q(X_y, \mathbb{C}).$$

Example Plane cubic curves.

$E_\lambda : y^2 z = x(x-z)(x-\lambda z)$, $\lambda \in \mathbb{C} \setminus \{0,1\}$.

Get family of smooth plane cubics over $\mathbb{C} \setminus \{0,1\}$.

Projection $(x,y,z) \longmapsto (x,z)$ gives $E_\lambda \to \mathbb{P}^1$, double cover ramified over $\{0, 1, \lambda, \infty\}$.

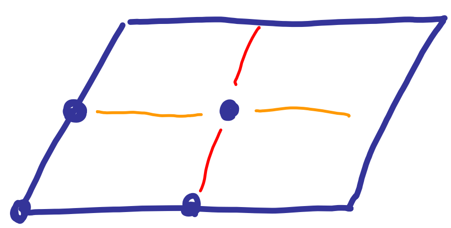


Under group law on cubic
$E_\lambda \cong \mathbb{C}/\Lambda$, have

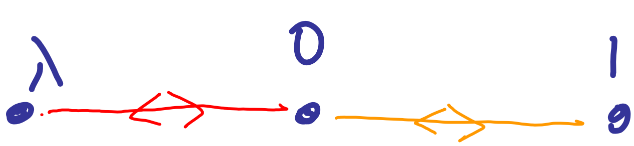
$E_\lambda / \{\pm 1\} \cong \mathbb{P}^1$,

(mult by -1 on $\mathbb{C}$, which descends to $\mathbb{C}/\Lambda$), and thus $0, 1, \lambda, \infty$ are identified w/ the 2-torsion points under the group law.

Have a basis of homology given by

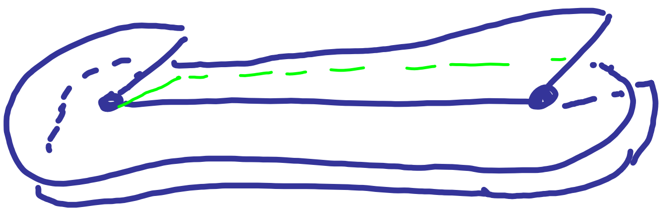


Images of these two loops in $\mathbb{P}^1$ look like:



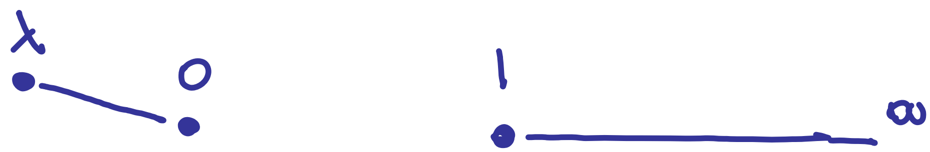
What does ramified double cover look like near these points?

single ramification point

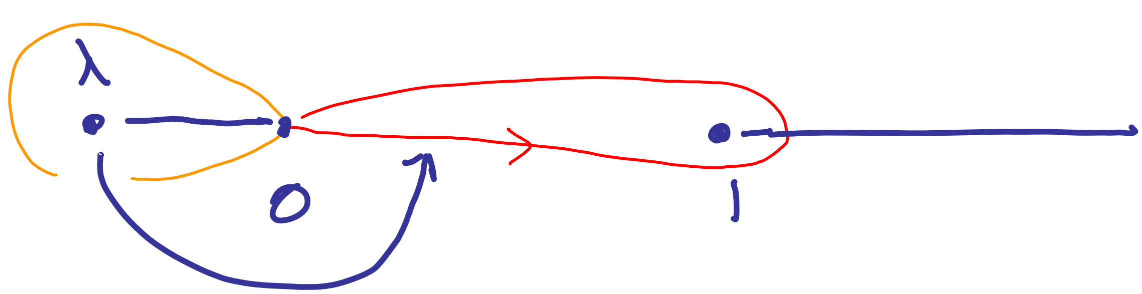


two points

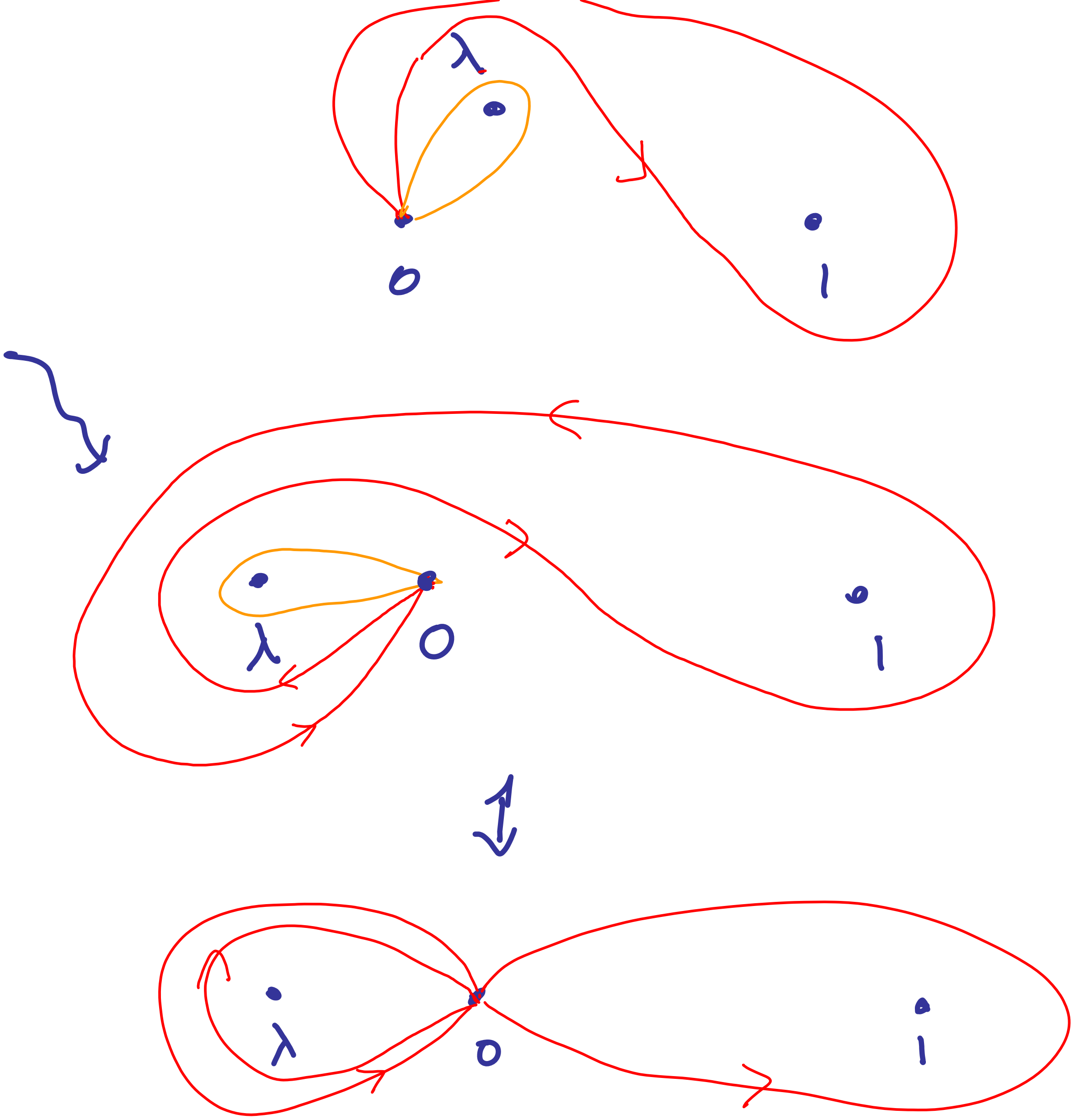
Now, consider λ in the region near zero:



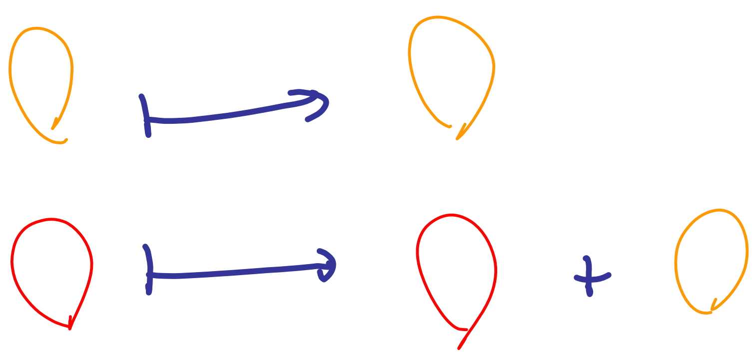
What happens as you move λ around 0?



$\rightsquigarrow$

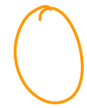


Fact



under monodromy.

[Hard for me to draw correct picture!]

Really undergoing Dehn twist about the vanishing cycle  .

Generally, monodromy <u>operator</u> μ given by

$$\mu(\alpha) = \alpha + \langle \alpha, \delta \rangle \delta$$

where δ is the vanishing cycle.

So, get info about nontriviality of family.

In same spirit,

<u>Variation of Hodge structure</u>

Given $\mathcal{X} \longrightarrow \Delta$ = disk in $\mathbb{C}$, proper & Kähler, get family of cohomologies $H^k(X_t, \mathbb{C})$. "Constant" by Ehresmann Thm. <u>But</u>,

$H^{p,q}(X_t)$ $(p+q=k)$ <u>vary</u> as $t \in \Delta$ varies.

[Choose "triv" of $H^k(\mathcal{X}, \mathbb{Z})$, possible, unique! Then get

$$H^k(X_t, \mathbb{C}) \underset{\text{canonical}}{=} H^k(\mathcal{X}, \mathbb{C})$$

$\forall t$.]

<u>Example 0</u> $\mathcal{X}$ a family of smooth proj curves, $k=1$. Get

$$H^{0,1}(X) \subset H^1(X, \mathbb{C})$$

complex subspace. Also have

$H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{C})$ mapping to $H^{0,1}(X)$.

We saw that image is a lattice in $H^{0,1}(X)$, and

$$H^{0,1}(X)/H^1(X,\mathbb{Z}) \cong \mathrm{Pic}^0(X) = \mathrm{Jac}(X),$$

the Jacobian of X. Comes equipped with a class Θ in

$$H^2(\mathrm{Jac}(X),\mathbb{Z}) \cong \Lambda^2 H^1(X,\mathbb{Z}),$$

intersection form! This makes $\mathrm{Jac}(X)$ a polarized abelian variety.

Torelli Thm X can be reconstructed from $(\mathrm{Jac}(X), \Theta)$.

Cor X can be reconstructed from

$$H^{0,1}(X) = V \subset H^1(X,\mathbb{C})$$

v.s. w/ $\mathbb{Z}$-structure.

Very Rough Idea of Pf [Andreotti-Frankel]:

- Gauss map of $\mathrm{Jac}(X)$ gives

$$\Theta^{\text{nonsing}} \longrightarrow \mathbb{P}(H^{0,1}(X)^*).$$

- Branch locus of this map is identified w/ projective dual of the canonically embedded copy of X!
- Note of course, only nonhyperelliptic case.