Recall the following, which made Hodge theory tick:

<u>Main Proposition</u> (Kähler Identities).

$(X, J, \langle,\rangle)$ a Kähler manifold. Then:

(1) $[\bar\partial, L] = 0 = [\partial, L]$, $[\bar\partial^*, \Lambda] = 0 = [\partial^*, \Lambda]$.

(2) $[\bar\partial^*, L] = i\partial$, $[\partial^*, L] = -i\bar\partial$,
$[\Lambda, \bar\partial] = -i\partial^*$, $[\Lambda, \partial] = i\bar\partial^*$.

(3) $\Delta_{\bar\partial} = \frac{1}{2}\Delta = \Delta_\partial$,

(4) Δ commutes with $\partial, \bar\partial, \partial^*, \bar\partial^*, L, \Lambda$.

<u>How to prove</u> First, on $\mathbb{C}^n$ with flat metric. Then $\omega = \frac{i}{2}\sum dz_k \wedge d\bar z_k$,

and for any form α,

$\bar\partial(\omega\wedge\alpha) = (\bar\partial\omega)\wedge\alpha + \cancel{(-1)^{\deg\omega}}\ \omega\wedge\bar\partial\alpha$
$= \omega\wedge\bar\partial\alpha$,

same for $\partial(\omega\wedge\alpha)$.

So $[\hat{\partial}, L] = 0$, $[\partial, L] = 0$.

In fact, same works on any Kähler mfld: $d\omega = 0$ and ω of type $(1,1)$ $\Rightarrow$ $\underbrace{\partial\omega}_{\text{type }(2,1)} + \underbrace{\bar{\partial}\omega}_{\text{type}(1,2)} = 0$

$\Rightarrow$ $\partial\omega = 0$, $\bar{\partial}\omega = 0$.

Second part of (1) is just adjoint of first part. This proves (1).

(2) requires actual proof. Again, on $\mathbb{C}^n$ w/ flat metric. let i_k, $i_{\bar{k}}$ denote contraction w/ $2\frac{\partial}{\partial z_k}$, $2\frac{\partial}{\partial \bar{z}_k}$ resp.

So e.g. $i_k(dz_k \wedge \alpha) = 2\alpha$. Then i_k adjoint to $dz_k \wedge -$, $i_{\bar{k}}$ adj to $d\bar{z}_k$.

Follows that

$$\Lambda = L^* = (\omega \wedge -)^* = -\frac{i}{2} \sum i_{\bar{k}} i_k .$$

Then integrating by parts, find

$$\partial^* \alpha = -\sum \frac{\partial}{\partial \bar{z}_k} i_k \alpha$$

$$\bar{\partial}^* \alpha = -\sum \frac{\partial}{\partial z_k} i_{\bar{k}} \alpha .$$

<u>Note</u> also that i_k and $i_{\bar{k}}$ anticommute with $dz_\ell \wedge -$ and $d\bar{z}_\ell \wedge -$ if $\ell \neq k$.

$[\, i_k (dz_\ell \wedge dz_k \wedge \alpha) = -i_k (dz_k \wedge dz_\ell \wedge \alpha) = -dz_\ell \wedge \alpha \,]$

Can then explicitly compute $[\Lambda, \bar{\partial}]$ and $[\Lambda, \partial]$ and compare to $-i\partial^*$, $i\bar{\partial}^*$ resp.

Example

$$\Lambda\bar\partial - \bar\partial\Lambda = -\frac{i}{2}\sum_k i_{\bar k} i_k \sum_\ell \frac{\partial}{\partial \bar z_\ell} d\bar z_\ell \wedge -$$

$$+ \frac{i}{2}\sum_\ell \frac{\partial}{\partial \bar z_\ell} d\bar z_\ell \wedge \Big(\sum_k i_k i_{\bar k} -\Big).$$

$$= \frac{-i}{2}\sum_k \Big[i_{\bar k} i_k \, d\bar z_k \frac{\partial}{\partial \bar z_k}(-) - d\bar z_k \, i_{\bar k} i_k \frac{\partial}{\partial \bar z_k}(-)\Big]$$

(since for $k \neq \ell$,

$$i_{\bar k} i_k d\bar z_\ell \wedge - = d\bar z_\ell \wedge (i_{\bar k} i_k -)$$

by note above).

$$= -i\Big(\sum_k i_k \frac{\partial}{\partial \bar z_k}(-)\Big) = -i\partial^*.$$

Similarly for the other one. This proves (2) on $\mathbb{C}^n$. Now, use first-order nature of identities plus "Kähler metric osculates flat metric to 2^{ND} order" to get (2) for any Kähler mfld.

(3). Now, use (2) to compute Laplacians! Note

$$i(\partial\bar{\partial}^* + \bar{\partial}^*\partial) = \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial = 0.$$

So $\partial\bar{\partial}^* = -\bar{\partial}^*\partial$. Similarly

$$\partial^*\bar{\partial} = -\bar{\partial}\partial^*.$$

Thus,

$$\Delta = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial})$$

$$= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$$

$$+ [\cancel{\bar{\partial}\partial^*} + \cancel{\partial\bar{\partial}^*} + \cancel{\partial^*\bar{\partial}} + \cancel{\bar{\partial}^*\partial}]$$

$$= \Delta_\partial + \Delta_{\bar{\partial}}.$$

Finally,

$$-i\Delta_\partial = \partial(\Lambda\bar\partial - \bar\partial\Lambda) + (\Lambda\bar\partial - \bar\partial\Lambda)\partial$$
$$= \partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial$$
$$= (\partial\Lambda - \Lambda\partial)\bar\partial + \bar\partial(\partial\Lambda - \Lambda\partial)$$
$$= -i\Delta_{\bar\partial}.$$

Done!

(4) similar, see Huybrechts. $\square$

Other deferred debt:

Hard Lefschetz

Let X be compact Kähler. Let

$$P^k(X,\mathbb{R}) = \ker(\Lambda : H^k(X,\mathbb{R}) \to H^{k-2}(X,\mathbb{R})).$$

$$P^{p,q}(X) = \ker(\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X)).$$

<u>Hard Lefschetz Theorem</u> X cpt Kähler of dim n. Then

(1) for $k \le n$,

$$L^{n-k} : H^k(X,\mathbb{R}) \to H^{2n-k}(X,\mathbb{R})$$

is an isomorphism.

(2) For any k,

$$H^k(X,\mathbb{R}) = \bigoplus_{i \ge 0} L^i P^{k-2i}(X,\mathbb{R}).$$

Moreover, both statements respect bidegree (so, $P^k(X,\mathbb{C}) = \bigoplus_{p+q=k} P^{p,q}(X)$) etc.).

Pf. L, Λ take harmonic forms to harmonic forms. So harmonic forms give a representation of $\mathfrak{sl}_2\mathbb{C}$.

Statement follows. $\square$

<u>Cor</u> $b_{i-2}(X) \leq b_i(X)$ for $i \leq n/2$.

<u>Recall</u>

$$Q(\alpha, \beta) = (-1)^{a(a-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{n-a},$$

$a = \deg \alpha$ ($= \deg \beta$, o.w. $Q(\alpha, \beta) = 0$).

<u>Prop</u> $Q(H^{p,q}(X), H^{p',q'}(X)) = 0$ unless $(p,q) \neq (q', p')$. Moreover, $i^{p-q} Q(\alpha, \bar{\alpha}) > 0$ for $\alpha \in P^{p,q}(X)$, $p+q \leq n$.