Line bundles

Let X be a complex manifold. A **holomorphic line bundle** on X is a complex manifold L with a holomorphic map $L \xrightarrow{\pi} X$ and a structure of complex 1-dim'l vector space on each fiber $\pi^{-1}(x)$ $(x \in X)$ with the property: for each $x \in X$ there is an open set $U \subseteq X$ with $x \in U$ and a holomorphic isomorphism

$$i_U : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}$$

such that

(1)
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "i_U"] \arrow[dr] & & U \times \mathbb{C} \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$
commutes

(2) the vector space structure on each $\pi^{-1}(y)$, $y \in V$, agrees with the one "induced" from $U \times \mathbb{C}$.

<u>Def</u> Let G be a complex Lie group.
A holomorphic principal G-bundle ~~over~~ over X is a complex manifold P equipped with a holo. map $P \xrightarrow{\pi} X$ and a G-action $G \times P \xrightarrow{a} P$ such that, for each $x \in X$, there is an open set $U \subseteq X$, $x \in U$, and a holomorphic isomorphism

$$\pi^{-1}(U) \underset{\sim}{\xrightarrow{i_U}} U \times G$$

satisfying

(1)
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "i_U"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$
commutes

(2) i_U is G-equivariant for the "obvious" G-action on $U \times G$ (by $h \cdot (y, g) = (y, hg)$).

<u>Example</u> In many instances, if G acts freely on X, $X \xrightarrow{\pi} X/G$ is a principal G-bundle. e.g.

$\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^n$ is a principal $\mathbb{C}^*$-bundle. [local trivializations, as they're called, are given by:

$$U_i = \{(x_0 : \dots : x_n) \mid x_i \neq 0\},$$

$$\pi^{-1}(U_i) = \{(x_0, \dots, x_n) \mid x_i \neq 0\},$$

$$\pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^*$$

$$(x_0, \dots, x_n) \longmapsto ((x_0 : x_1 : \dots : x_n), x_i).]$$

<u>Remark</u> If $P \to X$ is a principal G-bundle, then G acts freely on P (and properly ?).

<u>Construction</u> Given a principal G-bundle $P \to X$ and a <u>character</u> $G \xrightarrow{\lambda} \mathbb{C}^*$ (i.e. group homomorphism), define a G-action on $\mathbb{C} \times P$ by $h \cdot (c, g) = (c\lambda(h^{-1}), hg)$.

Then the <u>associated line bundle</u> $\mathbb{C} \times_G P$ is $\mathbb{C} \times P / G$.

<u>Prop</u> $\mathbb{C} \times P/G \longrightarrow P/G = X$ is a line bundle over X.

<u>Def</u> A <u>section</u> of a holomorphic map $X \xrightarrow{f} Y$ is a holomorphic map $\sigma : Y \longrightarrow X$ such that the composite $Y \xrightarrow{\sigma} X \xrightarrow{f} Y$ is the identity.

Example Consider $P = \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$.

One gets a line bundle for each $k \in \mathbb{Z}$:

Def $\mathcal{O}_{\mathbb{P}^n}(k) = \mathbb{C} \times P / \mathbb{C}^*$

where $\mathbb{C}^*$ acts on $\mathbb{C}$ via

$$\lambda(z) = z^{-k}.$$

Prop The set of sections of a line bundle $\mathbb{C} \times P/G \longrightarrow P/G$ is naturally in bijection with the set of functions $P \xrightarrow{\varphi} \mathbb{C}$ such that

$$\varphi(g \cdot y) = \lambda(g)^{-1} \varphi(y) \quad \forall g \in G,\ y \in P.$$

We call such functions λ-monodromic or λ-homogeneous

Proof. Let S denote the latter set of equivariant functions. Let

$H^0(X, L)$ denote the set of sections of $\mathbb{C} \times P/G =: L \longrightarrow P/G =: X$.

Given $\varphi \in S$, define $\tilde{\sigma}_\varphi$ as the composite

$$\begin{tikzcd} P \arrow[r, "(\varphi, id)"] \arrow[rr, bend right, "\tilde{\sigma}_\varphi"'] & \mathbb{C} \times P \arrow[r, "\pi"] & \mathbb{C} \times P/G. \end{tikzcd}$$

then $\tilde{\sigma}_\varphi(gy) = (\varphi(gy), gy)$

$$= (\lambda(g)^{-1}\varphi(y), gy) \sim (\varphi(y), y).$$

So $\tilde{\sigma}_\varphi(gy)$ is a G-invariant holomorphic map, hence it defines a holomorphic map

$$P/G \xrightarrow{\sigma_\varphi} L.$$

It's easy to check it's a section ...

Conversely, given a section

$$\sigma : P/G \longrightarrow \mathbb{C} \times P/G,$$

define a map $\varphi_\sigma : P \to \mathbb{C}$ by

$\varphi_\sigma(y) = c_y$ where

$\sigma(\pi(y)) = (c_y, y)$: that is,

$\sigma(\pi(y))$ is represented in $L = \mathbb{C} \times P / G$

by $(c_y, y) \in \mathbb{C} \times P$.

If we start from $\varphi \in S$, then

$\varphi_{\sigma_\varphi} = \varphi$ it's easy to see.

So $S \longrightarrow H^0(X, L)$ is injective.

To see that $S \to H^0(X, L)$ is surjective, it suffices to check that the function φ_σ produced by the above construction is holomorphic and λ-monodromic.

To see λ-monodromic, compute:

$$\sigma(\pi(gy)) = (c_{gy}, gy) \sim (c_y, y) = \sigma(\pi(y))$$

$$\Rightarrow (c_{gy}\lambda(g), g^{-1}gy) = (c_y, y)$$

$\Rightarrow \quad c_{gy}\lambda(g) = c_y. \Rightarrow \quad c_{gy} = \lambda(g)^{-1} c_y.$

So φ_σ is λ-monodromic.

Finally,

<u>Exercise</u> Check that φ_σ is holomorphic. $\square$.

<u>Example</u> Consider $\lambda : \mathbb{C}^* \to \mathbb{C}^*$

$$\lambda(z) = z^{-k}.$$

Then $\varphi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ is λ-monodromic if

$$\begin{aligned}\varphi(zx_0, \ldots, zx_n) &= \lambda(z)^{-1}\varphi(x_0, \ldots, x_n)\\ &= z^k \varphi(x_0, \ldots, x_n),\end{aligned}$$

i.e. φ is <u>homogeneous of degree k</u>.

<u>Claim</u> Such φ are exactly homogeneous polynomials of degree k in

$$\mathbb{C}[x_0, \dots, x_n].$$

Pf. If φ is holomorphic with domain $\mathbb{C}^{n+1} \setminus \{0\}$, then φ extends uniquely to a holomorphic function on $\mathbb{C}^{n+1}$ by Hartogs' Theorem.

Expanding in power series in a neighborhood of zero,

$$\varphi = \sum c_I x^I,$$

we get

$$\varphi(zx) = \sum c_I z^{|I|} x^I.$$

Then

$$\varphi(zx) - z^k \varphi(x) = 0$$

$$\Rightarrow \quad \sum c_I (z^{|I|} - z^k) x^I \equiv 0$$

$\Rightarrow \quad c_I = 0$ when $|I| \neq k$. This proves the claim. $\square$

Summarizing:

Prop $H^0(\mathbb{P}^n, \mathcal{O}(k)) \cong \mathbb{C}[x_0, \dots, x_n]_k$

homogeneous polys of deg k.

Divisors

Def Let X be a complex manifold.
An <u>analytic hypersurface</u> $V \subseteq X$ is a closed subset that is locally given as the zero set of a single nonzero holomorphic function.

It is <u>irreducible</u> if it is not the union of two proper closed subsets that are analytic hypersurfaces.

A <u>divisor</u> D on X is a formal finite linear combination $D = \sum a_i [Y_i]$

where each φ_i is an irreducible analytic hypersurface of X.

<u>Important Fact</u> The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ (a.k.a. convergent power series) is a UFD.

[Follows from Weierstrass Preparation Thm — see Huybrechts]

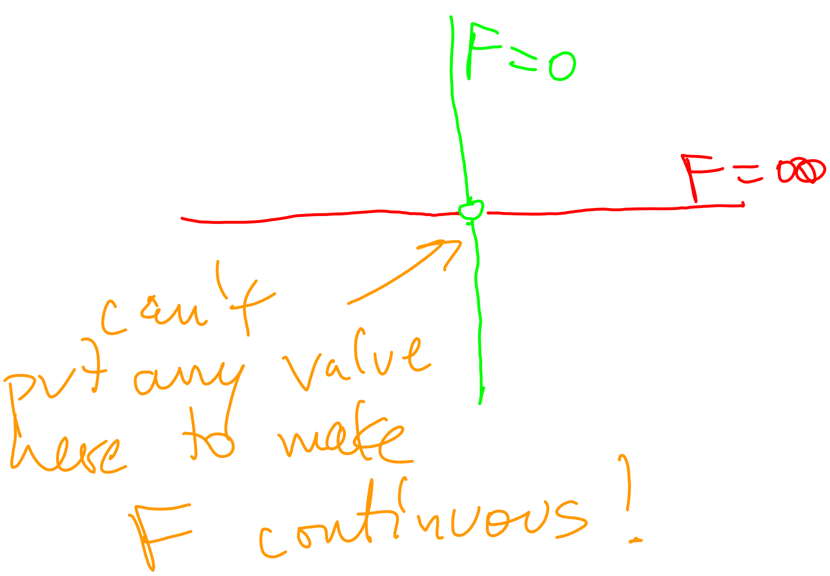
Suppose $x \in X$, $V \subset X$ is a hypersurface containing x defined by a germ $g \in \mathcal{O}_{X,x}$. Then V is <u>locally irreducible</u> at x iff g can be chosen irreducible.

<u>Recall</u> A <u>meromorphic function</u> F on an open set $U \subset X$ is a function defined on a dense open set $V \subset U$ such that, for every $p \in V$, $\exists$ a neighborhood U_p of p

in V and holomorphic functions f, g on W_p such that $F|_{W_p} = \frac{f}{g}$.

<u>Remark</u> Note that, unlike in <u>one</u> complex variable, there is no sensible notion for a meromorphic function (in general) of "$F(p) = \infty$"! Indeed, let

$$F(x,y) = \frac{x}{y}.$$ For $y \neq 0$ it makes sense to say $F(0,y) = 0$. For $x \neq 0$ it makes sense to say $F(x,0) = \infty$. But at $(x,y) = (0,0)$; no good:



Def Let $V \subset X$ be a hypersurface, $x \in V$. If V is locally irreducible at x, defined by irreducible holo. fn. g defined near x, and F, G are holomorphic functions defined near x, then

$$\operatorname{ord}_{V,x}(F/G) = \operatorname{ord}_{V,x}(F) - \operatorname{ord}_{V,x}(G)$$

where $\operatorname{ord}_{V,x}(H) = k$ if

$$H = g^k \cdot h \quad \text{where } h \text{ is holomorphic}$$

near x and $h(x) \neq 0$.

Fact By UFD property, this is well-defined and only depends on the meromorphic function F/G (not the specific numerator and denominator).

$\operatorname{ord}_{V,x}$ is the "order of vanishing along V at x."

If V is an irreducible analytic hypersurface, there exists $p \in V$ such that V is locally irreducible at p, and then we define

$$\mathrm{ord}_V (F/G) = \mathrm{ord}_{V,p} (F/G).$$

This gives the same answer for any p at which it is defined (and such p exist).

Let X be a compact complex manifold.

<u>Def</u> Suppose s is a nonzero holo. section of a holomorphic line bundle $L \xrightarrow{\pi} X$. We define a divisor $\mathrm{div}(s)$ as follows: the multiplicity of an irreducible hypersurface $V \subset X$ in $\mathrm{div}(s)$ is $\mathrm{ord}_V(\tilde{s})$ where $\tilde{s}$ is a function defined as follows: given $p \in V$ at which V is locally irreducible, find a local

trivialization $i_U : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}$ of L over a nbhd U of p. Then

$$i_U \circ s|_U = (\mathrm{id}_U, \tilde{s}).$$

<u>Fact</u> The divisor $\mathrm{div}(s)$ so defined doesn't depend on any of the choices made.

<u>Example</u> Let $s \in H^0(\mathbb{P}^n, \mathcal{O}(k)) = \mathbb{C}[x_0, .., x_n]_k$.

Factor s into irreducible polynomials,

$s = s_1 s_2 \dots s_r$. One can check that they must be homogeneous. Let

$$\mathrm{div}(s_i) = \{ \ell \in \mathbb{P}^n \mid s_i(\ell) = 0 \}.$$

Then

$$\mathrm{div}(s) = \sum \mathrm{div}(s_i).$$

[Discuss!!]