From line bundles to divisors

Let s be a meromorphic section of a line bundle $L \xrightarrow{\pi} X$. (i.e. $\exists$ open cover $\{U_i\}$ of X and trivs, $\pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}$ in which s is identified with a mero. fn — equivalently, in every local triv!).

Get a divisor $\mathrm{div}(s)$: if $V \subset X$ is an irred. hypersurface, choose $x \in V$ at which V is locally irreducible and such that s is defined (possibly $s(x) = \infty$) at x. Choose local triv. of L on nbhd U of x, $\pi^{-1}(U) \underset{\varphi}{\cong} U \times \mathbb{C}$. Then

$\varphi \circ s = (id, F)$ for a mero. fn F.

Def Multiplicity of V in $\mathrm{div}(s)$ is $\mathrm{ord}_V(F)$.

Lemma Doesn't depend on choices.

Reason If F, F' are the mero. fns obtained from s in two different trivializations φ, φ', then, writing

$$\varphi' \circ \varphi^{-1}(y, c) = (y, g(y)\cdot c)$$

where $y \in V$, $c \in \mathbb{C}$, we have

$$F' = g \cdot F.$$

[Pf. $\varphi \circ s = (id, F)$, $\varphi' \circ s = (id, F')$,

so $\varphi' \circ \varphi^{-1}(\varphi \circ s) = \varphi' \circ s = (id, F')$

$\underset{\displaystyle (id, g\cdot F)}{\parallel}$]

Since g is <u>nonvanishing</u> holo fn,

$$\mathrm{ord}_v(F) = \mathrm{ord}_v(g\cdot F) = \mathrm{ord}_v(F'). \quad \square$$

<u>Def</u> Two divisors D, D' on X are <u>linearly equivalent</u> if $D - D' = \mathrm{div}(F)$ for some meromorphic function F on X.

$\mathrm{div}(F)$ is a <u>principal divisor</u>.

<u>Prop</u> If s, s' are two meromorphic sections of L, then $\mathrm{div}(s)$ and $\mathrm{div}(s')$ are linearly equivalent.

Pf. Define F as follows: cover X by opens U_i with trivializations

$$\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}.$$

Write $\varphi_i \circ s = (\mathrm{id}, s_i)$, $\varphi_i \circ s' = (\mathrm{id}, s_i')$.

Let $F = \dfrac{s_i}{s_i'}$ on U_i.

<u>Claim</u> $F|_{U_i} = F|_{U_j}$ on $U_i \cap U_j$.

Pf. Recall that have <u>transition fns</u> g_{ij} given by

$$\varphi_i \circ \varphi_j^{-1}(u, c) = (u, g_{ij} \cdot c) \quad \text{for } u \in U_i \cap U_j,\ c \in \mathbb{C}.$$

Then:

$$\underbrace{\varphi_i \varphi_j^{-1} \varphi_j s}_{\varphi_i s = (\mathrm{id}, s_i)} = \varphi_i \varphi_j^{-1}(\mathrm{id}, s_j) = (\mathrm{id}, g_{ij} s_j)$$

$\varphi_i s = (\mathrm{id}, s_i)$, so

$$\boxed{s_i = g_{ij} s_j.}$$

Thus

$$\frac{s_j^i}{s_j'^i} = \frac{g_{ij}s_j}{g_{ij}s_j'} = \frac{s_i}{s_i'}. \qquad \square$$

Finishing proof of prop, have

$$s = F \cdot s'.$$

Now $\operatorname{div}(s) = \operatorname{div}(F) + \operatorname{div}(s')$. $\square$.

Let $\operatorname{Div}(X)$ denote group of divisors. Let $\operatorname{Prin}(X)$ denote its subgroup of principal divisors. Let $\widetilde{\operatorname{Pic}}(X)$ denote set of isom. classes of line bundles on X that have nonzero sections.

Get a map

$$\widetilde{\operatorname{Pic}}(X) \xrightarrow{\operatorname{div}} \operatorname{Div}(X)/\operatorname{Prin}(X)$$

by $\quad L \longmapsto \operatorname{div}(s)$ for some nonzero section s of L.

We'll see:

(1) div is surjective.

(2) div is injective.

(3) If we give $\widetilde{\mathrm{Pic}}(X)$ a group structure by $\otimes$, div is a group homom.

Later:
(4) If X is projective, $\widetilde{\mathrm{Pic}}(X) = \mathrm{Pic}(X)$ when $\mathrm{Pic}(X)$ denotes set of isom. classes of <u>all</u> holomorphic line bundles.

OK, so:

(1) Given divisor D, we can write D <u>locally</u> as $\mathrm{div}(g_i)$ on open set U_i so that $\{U_i\}$ covers X. let

$$g_{ij} = g_i/g_j \quad \text{on} \quad U_i \cap U_j.$$

<u>Note</u> • $g_{ij}g_{jk} = g_i/g_j \cdot g_j/g_k = g_i/g_k = g_{ik}$
on $U_i \cap U_j \cap U_k$.

• $g_{ii} = 1$.
• $g_{ij} \in \mathcal{O}(U_i \cap U_j)^*$, i.e. nonvanishing, and holo.

<u>Fact</u> Given any open cover $\{U_i\}$ of X

and $g_{ij} \in \mathcal{O}(U_i \cap U_j)^* \ \forall i,j$ s.t.

- $g_{ii} = 1 \quad \forall i,$
- $g_{ij} g_{jk} = g_{ik} \quad \forall i,j,k$,

there is a line bundle L on X, & choice of local trivs. φ_i over U_i, such that $\varphi_i \circ \varphi_j^{-1} = (id, g_{ij}) \ \forall i,j$.

<u>Rmk</u> If we start from L, produce $\{g_{ij}\}$, and glue to get L', then $L \cong L'$.

<u>(Sort of) more general</u> Given complex manifolds U_i with open sets

$$U_{ji} \subset U_i \quad \forall i,j \quad \text{and isoms.}$$

$$\varphi_{ij} : U_{ij} \to U_{ji} \quad \forall i,j$$

such that $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ on their domains,

$\exists$ a "complex space" X obtained by gluing U_i along U_{ij}, i.e. as colimit of $\coprod U_{ij} \rightrightarrows \coprod U_i$.

Warning Need not be a manifold.

Back to our story: define s locally by $s|_{U_j} = (id, g_j)$ in local triv. φ_j. Then

$$\varphi_i \circ \varphi_j^{-1} \circ s|_{U_j} = (id, g_{ij} \cdot g_j) = \left(id, \frac{g_i}{g_j} g_j\right) = (id, g_i)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad // \quad s|_{U_i}$$

So these local sections all agree, give mero. section s of L. Finally, clear that $\operatorname{div}(s) = \operatorname{div}(g_j) = D$ locally, hence globally. This proves (i).

We'll prove (3) next. It mainly depends on understanding the tensor product and duals of line bundles.

Main point If L, M are line bundles with transition functions $\{g_{ij}\}$, $\{h_{ij}\}$, then $L \otimes M$ has trans. fns $\{g_{ij} h_{ij}\}$, L^* has trans. fns $\{1/g_{ij}\}$.

Then if $\{s_i\}$, $\{t_i\}$ define sections of L, M respectively, then

$s_i t_i$ defines a section of $L \otimes M$,

$1/s_i$ defines a section of L^*.

<u>Cor</u> $\widetilde{\mathrm{Pic}}(X) \longrightarrow \mathrm{Div}(X)/\mathrm{Prin}(X)$ is a gp homom.

Pf. Follows from:

$\mathrm{div}(FG) = \mathrm{div}(F) + \mathrm{div}(G)$,

$\mathrm{div}(1/F) = -\mathrm{div}(F)$. for functions $\square$.

Finally,

(2) div is injective.

Pf. Since it's a group homom, suffices to show: if L is a line bundle and s is a nonzero mero. section with $\mathrm{div}(s) = 0$, then L is trivial. But $\mathrm{div}(s) = 0 \implies s$ is a nonvanishing holo.

section. This <u>defines</u> an isom.

$$X \times \mathbb{C} \longrightarrow L$$ of

line bundles. $\square$