Sheaves and Their Cohomology

We've already talked about sheaves of holomorphic functions. More examples:

(1) $X \xrightarrow{\pi} Y$ a (continuous, C^∞, holomorphic) map. Taking

$$Y \supseteq U \longmapsto \{\text{sections of } \pi^{-1}(U) \to U \text{ (continuous, } C^\infty\text{, holo.)}\}$$

is a sheaf, <u>sheaf of sections</u>.

(2) Let G be an abelian group. Get a <u>presheaf</u> (i.e. functor)

$$\mathrm{Top}(X)^{op} \longrightarrow \underline{\mathrm{Groups}}$$

$\uparrow$ category of open subsets of X, with inclusions as maps

by $\quad U \longmapsto G$. This is the <u>constant presheaf</u>. It may <u>not</u> be a sheaf.

Dumb example: $X = \{p,q\}$. Then

$$\{p,q\} \longmapsto G,$$
$$\{p\} \longmapsto G,$$
$$\{q\} \longmapsto G,$$

so sections $g_1 \in G$ over p, $g_2 \in G$ over q <u>don't</u> glue to a global $g \in G$ over $\{p,q\}$.

<u>Sheafification</u> The <u>stalk</u> of a sheaf $\mathcal{F}$ at $p \in X$ is

$$\mathcal{F}_p = \varinjlim_{p \in U \subset X} \mathcal{F}(U).$$

If $\mathcal{F}$ is a presheaf, sheafification $\mathcal{F}^+$ is given by

$$\mathcal{F}^+(U) = \left\{ \text{functions } f : U \longrightarrow \coprod_{p \in U} \mathcal{F}_p \;\middle|\; \begin{array}{l}(1),\\(2)\\\text{hold}\end{array} \right\}$$

(1) $f(p) \in \mathcal{F}_p \ \forall p \in U$

(2) $\forall p \in U \ \exists$ nbhd $p \in V \subset U$ and $s \in \mathcal{F}(V)$ s.t. $f(q)$ is image in $\mathcal{F}_q$ of s for <u>all</u> $q \in V$

(f is locally a section of $\mathcal{F}$).

Prop $\mathcal{F}^+$ is a sheaf.

Example Constant presheaf $\mathcal{F}$ associated to ab. gp G. Then

$$\mathcal{F}^+(U) = \prod_{C} G$$

where C is the set of connected components of U.

Remark $\mathcal{F}^+$ has a universal property... see e.g. Hartshorne Ch. II §1.

On to sheaf cohomology. First, Čech cohomology. Given an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X, define

$$U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}.$$

Suppose I is well-ordered. Let

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p}).$$

Want to define maps $C^p \xrightarrow{d_p} C^{p+1}$.

Suppose $\alpha \in C^p$, so $\alpha_{i_0 \dots i_p} \in \mathcal{F}(U_{i_0 \dots i_p})$.

Define $d_p\alpha$ so that

$$(d_p\alpha)_{i_0 \dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \hat{i_k} \dots i_{p+1}}\Big|_{U_{i_0 \dots i_{p+1}}}.$$

omit!

<u>Fact</u> $d_{p+1} \circ d_p = 0$.

Thus, get a <u>complex</u>

$$0 \to C^0 \to C^1 \to C^2 \to \dots$$

<u>Def</u> The <u>Čech cohomology</u> $\check{H}^p(\mathcal{U}, \mathcal{F})$ is

$$\frac{\ker d_p}{\operatorname{im} d_{p-1}}.$$

[Note: Naturally get a cosimplicial abelian group and this is associated complex of groups.]

<u>Example</u> $\mathcal{F} = \mathcal{O}^*_X$, X a cplx manifold.

$$C^1(\mathcal{U}, \mathcal{O}^*) = \prod_{i<j} \mathcal{O}^*(U_i \cap U_j).$$

So, an element $g \in C^1(\mathcal{U}, \mathcal{O}^*)$ is $g = (g_{ij})$. Now "Σ" in dg means <u>product</u> ($\mathcal{O}^*$ means nonvanishing holo. fns with <u>product</u> as group operation),

so for $i<j<k$,

$$(dg)_{ijk} = g_{jk} g_{ik}^{-1} g_{ij},$$

and $g \in \ker d_1$ means

$$g_{ik} = g_{ij} g_{jk}. \quad \text{Look familiar?}$$

<u>Note</u> To identify (g_{ij}) with transition functions, let $g_{ji} = 1/g_{ij}$.

Prop For a sufficiently nice open cover $\mathcal{U}$ of X,

$$\ker d_1 \longrightarrow \left\{ \begin{array}{c} \text{isom classes of} \\ \text{line bundles} \end{array} \right\}$$

is surjective and has kernel $\operatorname{im} d_0$.

For example, a covering by open balls in $\mathbb{C}^n$ works.

In other words, the group of isomorphism classes of line bundles is isomorphic to $\check{H}^1(\mathcal{U}, \mathcal{O}^*)$ in such a case.

More canonical Form

$$\varinjlim_{\mathcal{U}} \check{H}^j(\mathcal{U}, \mathcal{F}),$$

colimit over open coverings.

$\boxed{\text{Painful to try to compute!}}$

Now, one can take Čech cohomology also of a more complicated gadget.

Given a <u>complex</u> of sheaves,

$$0 \to \mathcal{F}_0 \xrightarrow{\delta_0} \mathcal{F}_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{k-1}} \mathcal{F}_k \to 0$$

and an open cover $\mathcal{U} = \{U_\alpha\}$, one gets a <u>double complex</u> of Čech complexes:

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{U}, \mathcal{F}_0) \arrow[r, "\delta_0^0"] \arrow[d, "d_0^0"] & C^0(\mathcal{U}, \mathcal{F}_1) \arrow[r] \arrow[d, "d_0^1"] & \cdots \\
0 \arrow[r] & C^1(\mathcal{U}, \mathcal{F}_0) \arrow[r, "\delta_0"] \arrow[d] & C^1(\mathcal{U}, \mathcal{F}_1) \arrow[r] \arrow[d] & \cdots \\
& \vdots & \vdots &
\end{tikzcd}$$

The <u>total complex</u> is $(C^\bullet, d_\bullet)$ given by

$$C^k = \bigoplus_{i+j=k} C^i(\mathcal{U}, \mathcal{F}_j),$$

$$d_k\big|_{C^i(\mathcal{F}_j)} = \delta_k + (-1)^j d_i^j .$$

Claim It's a complex.

Now, cohomology (in the absolute sense) is defined by combining a "good" open covering with a "good" resolution

$$\mathcal{F} \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_k$$

of $\mathcal{F}$; i.e. a complex

$$0 \to I_0 \to I_1 \to \cdots \to I_k \to 0$$

s.t.

$$0 \to \mathcal{F} \to I_0 \to I_1 \to \cdots \to I_k \to 0$$

is exact, and each I_ℓ is well-behaved for the given open covering.

We'll spell out next time what that means.

<u>But</u>

<u>Example 1</u> $\mathcal{F}$ = sheaf of holo. sections of a holo. vector bundle, $\mathcal{U}$ a cover by open balls. Then $\mathcal{F}$ is already good for $\mathcal{U}$!

<u>Example 2</u> $\mathcal{F}$ as above. We'll get the <u>Dolbeault complex</u>

$$\Omega^{0,0}(\mathcal{F}) \xrightarrow{\bar{\partial}} \Omega^{0,1}(\mathcal{F}) \longrightarrow \cdots$$

This is good for $\mathcal{U} = \{X\}$.