Math 511, Exam 1

Rules: Exam is due on Friday at the beginning of class (you may also turn it in before then). You may *not* discuss the exam with *anyone* else except Nevins—this means in person, via telephone, via the internet, or in any other manner. You may refer to results proven in Hartshorne without proving them, but you should supply proofs if you wish to use the results of any homework exercises. I am happy to answer questions about the exam policy.

Justify your answers to all problems!

Problem 1. Let k be a field.

(1) Let $\mathbf{A}^1_k = \operatorname{Spec}(k[x])$, $P = (x) \in \mathbf{A}^1_k$, $U = \mathbf{A}^1_k \setminus \{P\}$. The subset U of $\mathbf{A}^1_k$ is open, and we let $j : U \to \mathbf{A}^1_k$ denote the canonical open immersion of schemes. Compute the stalk $(j_*\mathcal{O}_U)_P$.

(2) Let $\mathbf{A}^2_k = \operatorname{Spec}(k[x,y])$, $Q = (x,y) \in \mathbf{A}^2_k$, and let $V = \mathbf{A}^2_k \setminus \{Q\}$, an open subset of $\mathbf{A}^2_k$. Let $i : V \to \mathbf{A}^2_k$ denote the canonical open immersion of schemes. Compute the stalk $(i_*\mathcal{O}_V)_Q$.

Problem 2. Let $X = \operatorname{Spec}(\mathbb{C}[x,y]/(xy)) \subset \mathbf{A}^2_{\mathbb{C}}$ be the closed subscheme defined by the equation $xy = 0$. For which $p \in X$ is the stalk $\mathcal{O}_{X,p}$ an integral domain? Use a picture to explain why the answer should be this.

Problem 3. Let $f(x,y) = y^2 - x^3 - x - \lambda \in \mathbb{C}[x,y]$, $\lambda \in \mathbb{C}$. Let $g(x,y) = xy$. Consider the curves Σ_f, Σ_g in $\mathbf{A}^2_{\mathbb{C}}$ defined by $\Sigma_f = \operatorname{Spec}(\mathbb{C}[x,y]/(f))$, $\Sigma_g = \operatorname{Spec}(\mathbb{C}[x,y]/(g))$.

Recall that the *scheme-theoretic intersection* of two closed subschemes $Z, W \subset X$ defined by quasicoherent sheaves of ideals $I_Z \subset \mathcal{O}_X$, $I_W \subset \mathcal{O}_X$ is the closed subscheme $Z \cap W$ corresponding to the sheaf of ideals $I_Z + I_W \subset \mathcal{O}_X$.

For which values of $\lambda \in \mathbb{C}$ is the scheme-theoretic intersection $\Sigma_f \cap \Sigma_g$ a reduced scheme?

Problem 4. Let X be a scheme locally of finite type over an algebraically closed field k (that is, equipped with a morphism $\pi : X \to \operatorname{Spec}(k)$ that is locally of finite type). Construct a bijection between closed points $x \in X$ and morphisms of schemes $f : \operatorname{Spec}(k) \to X$ such that $\pi \circ f = 1_{\operatorname{Spec}(k)}$.

Problem 5. Fix the ground field $k = \mathbb{C}$. I gave an argument in class that the quasicoherent sheaf of ideals in $\mathcal{O}_{\mathbf{A}^2}$ associated to the ideal $(x,y) \subset \mathbb{C}[x,y]$ is not a locally free $\mathcal{O}_{\mathbf{A}^2}$-module. [Feel free to ask me for more details, or if you don't remember the argument.]

(1) By contrast, prove that for any closed point $p \in \mathbf{A}^1_{\mathbb{C}}$, the sheaf of ideals I_p associated to the closed subscheme $\operatorname{Spec}(k(p)) \subset \mathbf{A}^1_{\mathbb{C}}$ is a free $\mathcal{O}_{\mathbf{A}^1}$-module.

(2) Now consider $X = \operatorname{Spec}(\mathbb{C}[x^2, x^3])$; here $\mathbb{C}[x^2, x^3]$ denotes the $\mathbb{C}$-subalgebra of $\mathbb{C}[x]$ generated by the elements 1, x^2 and x^3. Is it true that for every closed point $p \in X$, the ideal I_p is a locally free $\mathcal{O}_X$-module?