Mathematics 511, Homework n

Hand in the first, second, fourth, and last problems.

(1) Let $\mathbf{M} \xrightarrow{\pi} \mathbb{P}^n$ be a line bundle over $\mathbb{P}^n$, with a choice of trivializations over the open sets $U_i \subset \mathbb{P}^n$ by bijections

$$\tilde{\phi}_i = (\phi_i, \Phi_i) : \pi^{-1}(U_i) \to \mathbb{A}^n \times k.$$

We can then form the transition functions

$$\psi_{ji} = \Phi_j \circ \Phi_i^{-1} : U_i \cap U_j \to \mathrm{GL}_1(k).$$

(Recall that, for each point $x \in U_i \cap U_j$, the map $\Phi_i(x)$ is a linear isomorphism from the fiber $\pi^{-1}(x)$ to k, so the composite $\Phi_j(x) \circ \Phi_i(x)^{-1}$ is a linear isomorphism from k to k, hence is given by scalar multiplication by a nonzero element of k.)

Prove that a collection of functions $F_i \in k[\mathbf{u}]$, $i = 0, \ldots, n$ as in the lectures, determine a regular section of the line bundle $\mathbf{M}$ by the procedure described in the lectures if and only if $F_j \circ \phi_j = (\psi_{ji}) \cdot (F_i \circ \phi_i)$ on $U_i \cap U_j$ for all i, j. Here $\phi_i : U_i \to \mathbb{A}^n$ is the usual coordinate chart on the open set U_i of $\mathbb{P}^n$.

(2) Prove that if $\mathbf{L} \xrightarrow{\pi} \mathbb{P}^n$ is a line bundle, then the transition functions for the dual line bundle

$$\mathbf{L}^* = \{(x, v) \mid x \in \mathbb{P}^n, v \in \mathrm{Hom}_k(\pi^{-1}(x), k)\}$$

are given by $\psi^*_{ji} = 1/\psi_{ji}$, where ψ_{ji} are the transition functions for $\mathbf{L}$.

(3) Prove that if $\mathbf{L} \xrightarrow{\pi} \mathbb{P}^n$ is a line bundle, with transition functions ψ_{ji}, then the transition functions for the line bundle

$$\mathbf{L}^{\otimes m} = \{(x, v) \mid x \in \mathbb{P}^n, v \in (\pi^{-1}(x))^{\otimes m}\}$$

are given by ψ^m_{ji}.

(4) Prove that in the scheme $\mathbb{P}^1_k = \mathrm{Proj}(k[x_0, x_1])$ with k a field, the homogeneous ideals (x_0) and $(x_0^2, x_0 x_1)$ define the same (one-point, reduced) closed subscheme. [Hint: find an open set of the form $D(f)$ that contains this closed subscheme, then calculate the two relevant localizations.]

(5) Suppose that $f \in k[x_0, x_1]$ is a (not necessarily irreducible) nonzero homogeneous polynomial of degree d. What is the dimension of the degree N component of $k[x_0, x_1]/(f)$ for N large? How could you interpret this in terms of the corresponding closed subscheme of $\mathbb{P}^1_k$? Describe the closed subscheme in the case $f = x_0^5$.

(6) Do problems 2.10, 2.11, and 2.12 from Chapter I, section 2 of Hartshorne.