Math 427, Midterm 1 **Name**_____

Note: there are more problems here than I expect you can do in the given time!

Problem 1. (6 points) Complete the definition: a subgroup H of a group G is *normal* if

Problem 2. (6 points) Prove that the kernel of a homomorphism $\phi : G \to H$ is a normal subgroup of G.

Problem 3. (8 points) Prove that there is no nontrivial homomorphism from $\mathbb{Z}/16\mathbb{Z}$ to $\mathbb{Z}/5\mathbb{Z}$.

Problem 4. (8 points) Give a complete classification of the homomorphisms from $\mathbb{Z}$ (which is a group under addition of elements) to $\mathbb{Z}$. Justify your answer.

Problem 5. (4+4+4+4 points) Let G be a group and let H and N be subgroups of G. Suppose that N is a normal subgroup of G.

(1) Prove that the set
$$HN = \{hn \mid h \in H, n \in N\}$$
is a subgroup of G that contains N.

(2) Prove that N is a normal subgroup of HN.

(3) Prove that the function $\phi : H \to HN/N$ defined by $\phi(h) = hN$ is a surjective homomorphism.

(4) What is the kernel of ϕ? Justify your answer.

Problem 6. (6 points) Does there exist a nonabelian group G of order 18 that has a nontrivial homomorphism to $\mathbb{Z}/3\mathbb{Z}$? Justify your answer.

Problem 7. (6 points) Recall that the *center* of G is

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}.$$

Recall that $Z(G)$ is always a subgroup of G.

Prove that any subgroup H of $Z(G)$ is a normal subgroup of G.

Problem 8. (8+ 4 + 2 points)

(1) Suppose that there is a subgroup H of G for which $H \leq Z(G)$ and G/H is cyclic. Prove that G must be abelian.

(2) Prove that if G is a nonabelian group of order 8 then the center of G cannot contain an element of order 4.

(3) In part (2), can the center contain an element of order 2?

Problem 9. (6 + 6 points) Let G denote the group of bijective (that is, one-to-one and onto) functions from $\mathbb{Z}$ to $\mathbb{Z}$ with composition of functions as the group operation. Let

$$H = \{f \in G \mid f(n) = n \text{ for all } n \geq 0\}.$$

(1) Prove that H is a subgroup of G.

(2) Find an element g of G such that $gHg^{-1} \subset H$ but $gHg^{-1} \neq H$. Justify your answer.

Problem 10. (8 points) Recall that the dihedral group D_n of order $2n$ has elements x and y such that x has order 2, y has order n, and $xyx^{-1} = y^{-1}$. Let k be an integer satisfying $1 \leq k \leq n-1$. For which choices of n and k is the element y^3x an element of the subgroup of D_n generated by the set $\{y^k, x\}$? Justify your answer.

Problem 11. (10 points) Consider the group $G = S_4$, the symmetric group of bijections from the set $\{1, 2, 3, 4\}$ to itself. Recall that G has order 24. For which of the positive integers $n = 2, 3, 4, 6, 8, 12$ that divide the order of G does there exist a subgroup of G of order n? Justify your answer. [Note: I think the hardest to determine is probably 8. You may *not* apply Sylow's theorems!]

You may use this page for scratchwork or solutions to problems.

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