Math 427, Midterm 2 **Name**______________________________________

Problem 1. (6 points) Let G be a group and S a nonempty set. Suppose we are given an action $a : G \times S \to S$ of G on S. Let

$$H = \left\{ g \in G \;\middle|\; a(g, x) = x \text{ for all } x \in S \right\}.$$

Prove that H is a normal subgroup of G.

Problem 2. (10 points) Prove that a group of order 63 must be abelian.

Problem 3. (8 points) Determine the group of automorphisms of $G = \mathbb{Z}/8\mathbb{Z}$ (that is, not just the *set* A of automorphisms of G, but the structure of that set A as a group under composition as well). Justify your answer.

Problem 4. (10 points) Suppose that G is a group of order $|G| = p^n$ where p is a prime number and $n \geq 1$ is an integer. Prove, *without referring to any version of the Sylow theorems*, that G contains a normal subgroup of order p^{n-1}. [Hint: try an induction.]

Problem 5. (8 points) Consider the group D_4, thought of as a subgroup of $\mathrm{GL}(2, \mathbb{R})$ acting by rotations and reflections on the plane $\mathbb{R}^2$. The group D_4 then also acts on the subset

$$S = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$$

of four points in the plane (draw a picture!). Consider the set X of all subsets of S that consist of two elements from S—for example, one element of the set X would be the two-element subset $\{(1, 0), (0, -1)\}$ of S. What are the orbits of the action of D_4 on X? Justify your answer.

Problem 6. (8 points) Prove that if H is a normal subgroup of a finite group G and $|H| = p^k$ for some prime p and nonnegative integer k, then H is contained in every Sylow p-subgroup of G.

Problem 7. (10 points) Prove that if $n \geq 5$ is an integer, then the symmetric group S_n has no subgroup of index r where $2 \leq r < n$.

Problem 8. (10 points) Suppose that G is a finite abelian group that is not cyclic. Prove that there is some prime number p such that G contains a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Problem 9. (6 points) Let G be an abelian group. Let $\widehat{G}$ denote the set of all homomorphisms $\phi : G \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ denotes the group of all nonzero complex numbers (with multiplication as the group operation). Given $\phi_1, \phi_2 \in \widehat{G}$, we define $\phi_1 \cdot \phi_2$ to be the function from G to $\mathbb{C}^\times$ given by $(\phi_1 \cdot \phi_2)(g) = \phi_1(g)\phi_2(g)$ for all $g \in G$.

Prove that $\phi_1 \cdot \phi_2$ is a homomorphism and that this product makes $\widehat{G}$ into an abelian group.

Problem 10. (3+5 points) Keeping the notations of the previous problem, prove:

(1) If $G = \mathbb{Z}$, then $\widehat{G} = \mathbb{C}^\times$.

(2) if $G = \mathbb{Z}/n\mathbb{Z}$ for some integer $n > 1$, then $\widehat{G} \cong \mathbb{Z}/n\mathbb{Z}$.

Problem 11. (5+3+3+3 points) An *inner automorphism* of a group G is an automorphism $\phi : G \to G$ (that is, an isomorphism from the group G to itself) of the form $\phi(g) = xgx^{-1}$ for some $x \in G$.

(1) Prove that the set $\mathrm{Inn}(G)$ of inner automorphisms of G forms a subgroup of the group $\mathrm{Aut}(G)$ of all automorphisms of the group G.

(2) Prove that the function $\psi : G \to \mathrm{Inn}(G)$ defined by $\psi(x)(g) = xgx^{-1}$ for all $g \in G$ is a surjective group homomorphism.

(3) Prove that if the center $Z(G)$ of a group G satisfies $Z(G) = \{e\}$, then $G \cong \mathrm{Inn}(G)$.

(4) Give an example of a group G and an automorphism $\phi : G \to G$ that is *not* an inner automorphism.

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