

## Mathematics 427, Homework 7

- (1) (T)(\*) Prove the Second Isomorphism Theorem: if  $G$  is a group and  $H, N$  are subgroups of  $G$  such that  $N$  is normal in  $G$ , then  $H/(H \cap N)$  is isomorphic to  $HN/N$ .
- (2) (T)(\*) Consider the direct product  $G \times H$  of groups  $G$  and  $H$ . Prove that the subsets

$$\overline{G} = \{(g, e) \mid g \in G\}$$

and

$$\overline{H} = \{(e, h) \mid h \in H\}$$

are normal subgroups of  $G \times H$ , that  $\overline{G} \cap \overline{H} = \{e\}$ , and that  $G \times H = \overline{G}\overline{H}$ .

- (3) (T)(\*) Suppose that  $L$  is a group with normal subgroups  $G$  and  $H$  such that  $G \cap H = \{e\}$  and  $L = GH$ . Prove that the function  $\phi : G \times H \rightarrow L$  given by  $\phi(g, h) = gh$  is a group isomorphism.
- (4) (T) Prove that the subgroup  $B$  of upper triangular matrices in  $\text{GL}(n, \mathbb{C})$  is conjugate to the subgroup  $L$  of lower triangular matrices: that is, that there exists  $g \in \text{GL}(n, \mathbb{C})$  such that  $gBg^{-1} = L$ .
- (5) (G)(\*) Prove or disprove: a nonabelian simple group cannot act nontrivially on a set containing fewer than five elements.
- (6) (G) Prove the Bruhat decomposition: let  $G = \text{GL}(n, \mathbb{C})$ , let  $B$  denote the subgroup of upper triangular matrices. Then  $G = \cup_{w \in W_n} BwB$ , where

$$BwB = \{b_1wb_2 \mid b_1, b_2 \in B\}.$$

- (7) (G)(\*) The *normalizer* of a subgroup  $H$  in a group  $G$  is the set

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}.$$

Prove or disprove: the normalizer  $N_G(H)$  is always a normal subgroup of the group  $G$ .

- (8) (V) Let  $M_n$  denote the set of all  $n \times n$  matrices with complex coefficients. Let  $\text{GL}(n, \mathbb{C})$  act on  $M_n$  by  $a(g, L) = gLg^{-1}$ .
- (a) Give a classification of the orbits of this action on  $M_n$ . [Note: this is deliberately slightly vague. One way to do it would be to give a single element in each orbit! Hint: how is Jordan canonical form relevant?]
- (b)  $M_n$  is also a complex vector space. The function  $L \mapsto \text{trace}(L)$  is a linear function on  $M_n$ , i.e.

$$\text{trace}(L_1 + \lambda L_2) = \text{trace}(L_1) + \lambda \text{trace}(L_2)$$

for all  $L_1, L_2 \in M_n$  and  $\lambda \in \mathbb{C}$ . Moreover, it is an *invariant* function, i.e.  $\text{trace}(gLg^{-1}) = \text{trace}(L)$  for all  $g \in \text{GL}(n, \mathbb{C})$  and all  $L \in M_n$ . Prove that any invariant linear function on  $M_n$  is a scalar multiple of the trace function.