Math 427, Homework 13

Problem 1. Prove that every ideal of $\mathbb{Z}$ is of the form

$$(n) = \{nk \mid k \in \mathbb{Z}\}$$

for some $n \in \mathbb{Z}$. Prove that $\mathbb{Z}/(n)$ is an integral domain if and only if either $n = 0$ or n is a prime number (or its negative).

Problem 2. Let k be a field. Prove that if the polynomial $f(z)$ is not irreducible—that is, if f factors as a product of two polynomials each of degree at least 1—then $k[z]/(f)$ is not an integral domain.

Problem 3. Let $a \in \mathbb{C}$ be a complex number that is not real. Define a function $e_a : \mathbb{R}[z] \to \mathbb{C}$ by $e_a(f) = f(a)$. Prove that e_a is a surjective ring homomorphism. Prove that if $m(z) = (z-a)(z-\overline{a})$ where $\overline{a}$ denotes the complex conjugate of a, then $m(z) \in \mathbb{R}[z]$ and that the kernel of e_a is the ideal generated by $m(z)$.

Problem 4. Let $S \subseteq R$ be a nonempty subset of a commutative ring R with 1. Recall that the ideal generated by S is defined to be

$$(S) = \bigcap_{S \subset J} J$$

where the intersection is taken over all ideals J of R that contain S.

Let I be the subset of R consisting of all expressions of the form $a_1 s_1 + \cdots + a_k s_k$, where k is some positive integer, each s_i is an element of S, and each a_i is an element of R. Prove that I is an ideal of R and that $(S) = I$.

Problem 5. Let p be a prime number. Does the polynomial $f(z) = z^p - z \in (\mathbb{Z}/p\mathbb{Z})[z]$ factor? If so, how?

Problem 6. Let k be a field. Let $S = k[z^2, z^3]$ denote the subset of $k[z]$ consisting of polynomials $f(z)$ of the form $f(z) = c_0 + c_2 z^2 + \cdots + c_n z^n$ for some $c_0, c_2, \ldots, c_n \in k$—in other words, polynomials whose coefficient of z is zero. Prove that S is also a ring under the usual addition and multiplication. Furthermore, prove that S is isomorphic to the quotient ring $k[x,y]/(y^2 - x^3)$.

Problem 7. Let k be a field. Suppose that p and q are elements of $k[z]$. Prove that $(p) \subseteq (q)$ if and only if there exists $v \in k[z]$ such that $p = qv$.

Problem 8. Let n be a positive integer. Let $f(z) = z^n$. Prove that $S = \mathbb{C}[z]/(f)$ is a finite-dimensional complex vector space with basis $1, z, z^2, \ldots, z^{n-1}$. Prove that multiplication by z defines a linear operator on S. Compute the matrix of this operator in this basis. Justify your answer.

Problem 9. Let k be a field, and let $a = (a_1, \ldots, a_n) \in k^n$ be an n-tuple of elements of k. Prove that $(x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal of $k[x_1, \ldots, x_n]$ as follows:

(1) Prove that if k is a field, R is a commutative ring with 1, and $\alpha : k \to R$ is a homomorphism such that $\alpha(1) = 1$, then α is injective. [Hint: what could the kernel be?]

(2) Prove that in the special case $R = k[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n)$ above, the homomorphism $\alpha : k \to R$ that takes a to the coset of the constant polynomial $f(x_1, \ldots, x_n) = a$, is surjective.

Problem 10. Suppose that $I \subset \mathfrak{gl}(n, \mathbb{C})$ is a right ideal of the ring of $n \times n$ matrices with complex coefficients.

(1) Let $V = \{Mv \mid M \in I, v \in \mathbb{C}^n\}$. Prove that V is a complex vector subspace of $\mathbb{C}^n$.

(2) Let
$$J = \{M \in \mathfrak{gl}(n, \mathbb{C}) \mid Mv \in V \text{ for all } v \in \mathbb{C}^n\}.$$
Prove that J is a right ideal and that $I \subseteq J$.

(3) Finally, prove that in fact $I = J$. [Hint: prove that I must contain a matrix M whose image, i.e. column space, is all of V. Then prove that if N is any matrix in J, there is some matrix g such that $N = Mg$.]

Turn in written solutions to problems 2, 3, 6, 7, and 8.