Math 427, Homework 14

Problem 1. Prove that if D is a commutative ring with 1, then D is an integral domain if and only if the following holds: for all $a, b, c \in D$ with $a \neq 0$, if $ab = ac$ then $b = c$.

Problem 2. Prove that if R is a commutative ring, F is a field, and $\phi : F \to R$ is a homomorphism of rings, then either ϕ is injective or $\phi(x) = 0$ for all $x \in F$.

Problem 3. Suppose that I, J are ideals of a commutative ring R.

(1) Let $I + J = \{u + v \mid u \in I, v \in J\}$. Prove this is an ideal of R that contains I and J.

(2) Let

$$IJ = \left\{ r \in R \,\middle|\, \exists n \geq 1, x_1, \ldots, x_n \in I, y_1, \ldots, y_n \in J \text{ so that } r = \sum_{i=1}^{n} x_i y_i \right\}.$$

Prove that IJ is an ideal of R.

Problem 4. Let $\mathbb{C}$ denote the complex numbers. Consider the ring homomorphism $\phi : \mathbb{C}[x] \to \mathbb{C}[x, y]/(y - x^2)$ for which $\phi(f(x))$ is the coset $f(x) + (y - x^2)$ (where $(y - x^2)$ denotes the ideal generated by $y - x^2$). Is ϕ an isomorphism? Justify your answer.

Problem 5. Suppose R and S are commutative rings with 1, and that $\phi : R \to S$ is a ring homomorphism. Prove that if $P \subset S$ is a prime ideal, then

$$\phi^{-1}(P) = \{r \in R \mid \phi(r) \in P\}$$

is a prime ideal of R. Give an example to show that, even if P is a maximal ideal, $\phi^{-1}(P)$ need not be a maximal ideal. [Hint: use $\mathbb{Z}$ and $\mathbb{Q}$ as your rings.]

Problem 6. Recall the ring $k[z^2, z^3]$ from the last homework. It is a subring of $k[z]$ (here k is a field). Let $i : k[z^2, z^3] \to k[z]$ denote the inclusion homomorphism. Prove that i^{-1} defines a bijection between the set of maximal ideals of $k[z]$ and the set of maximal ideals of $k[z^2, z^3]$.

Problem 7. Let R be a commutative ring with 1. Suppose that R has dimension $n \geq 1$. Prove that if R is an integral domain and $I \subset R$ is a nonzero proper ideal, then R/I has dimension at most $n - 1$. Use this to conclude that a polynomial ring in n variables $k[x_1, \ldots, x_n]$, where k is a field, has dimension at least n. [In fact, its dimension is exactly n, but this is a bit harder to prove.]

Problem 8. Prove that if R is a commutative ring with 1 and $a, b \in R$, then $(a) = (b)$ if and only if there exists a unit $u \in R$ such that $a = bu$.

Problem 9. Let $S = \mathbb{Z}[i\sqrt{5}]$ be the subring of $\mathbb{C}$ defined by:

$$S = \mathbb{Z}[i\sqrt{5}] = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\}.$$

Prove that in the ring S ,

(1) $3 \cdot 2 = 6 = (1 + i\sqrt{5})(1 - i\sqrt{5})$.

(2) All the elements $3, 2, (1 + i\sqrt{5})$, and $(1 - i\sqrt{5})$ are irreducible. [Hint: What are the possible absolute values of the complex numbers in S? What happens to the absolute value when you factor?]

(3) The only units in S are 1 and -1.

Use these to prove that S is not a UFD.

Problem 10. For a positive integer n, let $\Gamma = \mu_n$ denote the group of nth roots of unity in $\mathbb{C}^\times$ (as a group under multiplication). Given $g \in \Gamma$ and $f(x, y) \in \mathbb{C}[x, y]$, let $a(g, f) = f(gx, g^{-1}y)$.

(1) Prove that a defines an action of the group Γ on the set $\mathbb{C}[x,y]$.
(2) Prove that for each $g \in \Gamma$, the function $\phi_g : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ defined by $\phi_g(f) = a(g,f)$ defines a ring isomorphism from $\mathbb{C}[x,y]$ to itself.
(3) Let
$$\mathbb{C}[x,y]^\Gamma = \{ f \in \mathbb{C}[x,y] \mid a(g,f) = f \text{ for all } g \in \Gamma \}.$$
Prove that $\mathbb{C}[x,y]^\Gamma$ is a subring of $\mathbb{C}[x,y]$.
(4) Compute $\mathbb{C}[x,y]^\Gamma$ in the special case of $\Gamma = \mu_2 = \{1,-1\}$. A harder problem: prove that the ring $\mathbb{C}[x,y]^{\mu_2}$ is isomorphic to $\mathbb{C}[u,v,w]/(uv - w^2)$.

Write up and turn in solutions to problems 3, 4, 5, 6, and 9.