## Mathematics 427, Homework 8

(1) Suppose that $G$ is a group with normal subgroups $G_1, G_2, \ldots, G_k$. Suppose further that for each $m$ satisfying $1 \le m \le k-1$, one has $G_1 \ldots G_m \cap G_{m+1} = \{e\}$. Assume also that $G = G_1 \ldots G_k$. Prove that $G \cong G_1 \times \cdots \times G_k$.

(2) Prove that if $G$ is a finite abelian group, then $G$ is isomorphic to the direct product of its Sylow subgroups.

(3) Let $n \ge 2$ be an integer. Let $G$ be a finite abelian group. We know that $G$ can be written, up to isomorphism, as a direct product of cyclic groups of prime power order:

$$G \cong \mathbb{Z}/p_1^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_n^{e_n}\mathbb{Z},$$

where $p_1, \ldots, p_n$ are prime numbers, each $e_i > 0$, and $|G| = p_1^{e_1} \cdot \cdots \cdot p_n^{e_n}$. Prove that $G$ is cyclic if and only if $p_i \neq p_j$ for $i \neq j$.

(4) How could you prove that the prime powers $p_i^{e_i}$ appearing in the last problem are invariants of the group $G$ (in the sense that if one has another such expression

$$G \cong \mathbb{Z}/q_1^{f_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/q_m^{f_m}\mathbb{Z},$$

then $n = m$ and, after re-ordering the primes $q_i$ as necessary, we have $p_i = q_i$ and $e_i = f_i$ for all $i$)?