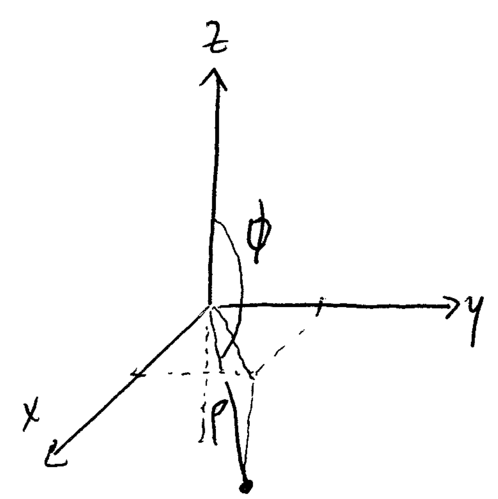
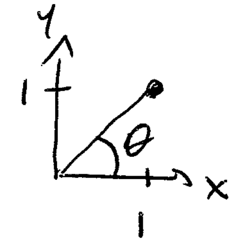
Problem 1

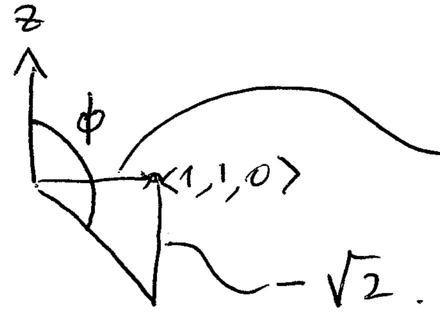
(1)





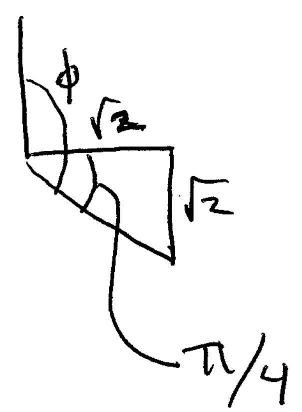
$\theta = \frac{\pi}{4}.$

$$\rho = \sqrt{1^2+1^2+(-\sqrt{2})^2} = \sqrt{1+1+2} = 2.$$



distance $= \sqrt{1^2+1^2} = \sqrt{2}.$

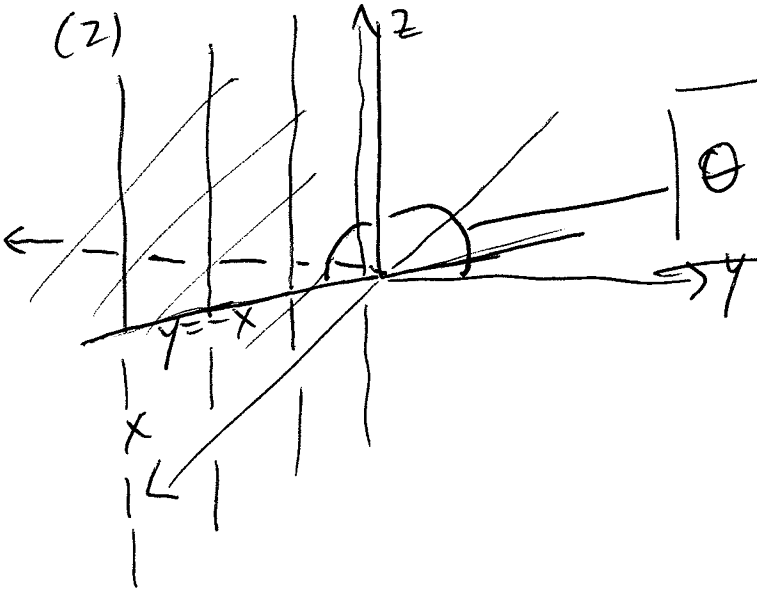
So,



So, $\phi = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.$

$$\boxed{(\rho, \phi, \theta) = \left(2, \frac{3\pi}{4}, \frac{\pi}{4}\right).}$$

(2)



$\boxed{\theta = \frac{7\pi}{4}}$.

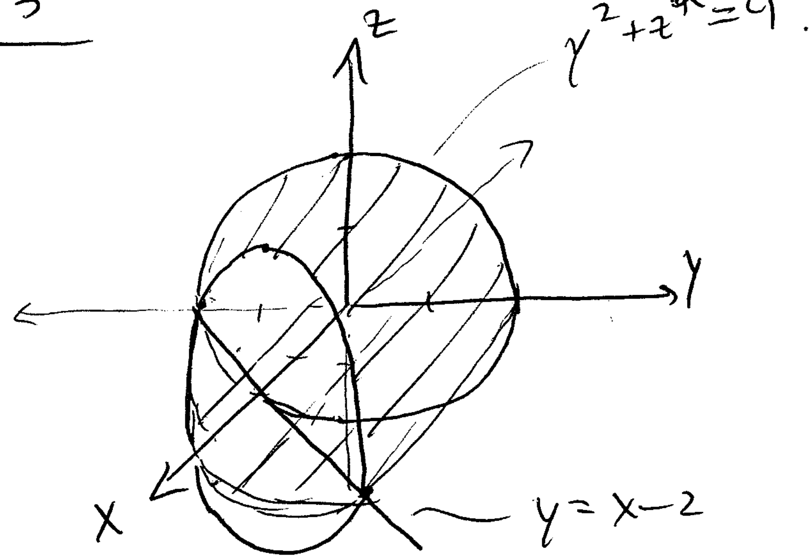
Problem 2

$$\int_{-2}^{1}\int_{2}^{4} x^2 y^3 \, dy \, dx = \int_{-2}^{1} x^2 \frac{y^4}{4}\Big|_2^4 \, dx$$

$$= \int_{-2}^{1} x^2 (64 - 4) \, dx = 60\frac{x^3}{3}\Big|_{-2}^{1} = 20 - \frac{60}{3}(-2)^3$$

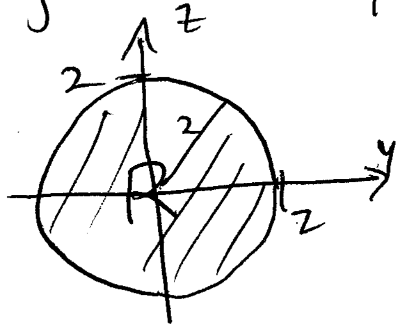
$$= 20 - 20(-8) = \boxed{180}$$

Problem 3



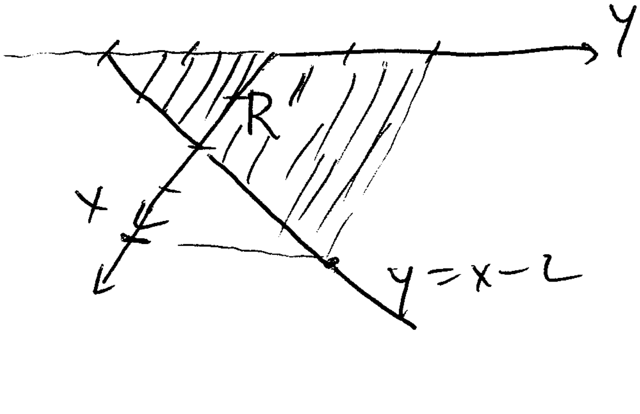
Can either:

(i) project to (y,z)-~~axis~~ plane. Solid lies over



and between the graphs of $x=0$, $x=y+2$.

(ii) project to (x,y)-plane. Solid lies over



between graphs of

$z=-\sqrt{4-y^2}$

and

$z=\sqrt{4-y^2}$.

Option (i) looks easier. Let's change the coordinates to cylindrical coords, using

$$y = r\cos\theta, \quad z = r\sin\theta, \quad x = x.$$

Then $\boxed{dx\,dy\,dz = r\,dr\,d\theta\,dx}$,

and R is defined by $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

Also, $0 \le x \le y + 2. = r\cos\theta + 2$

So,

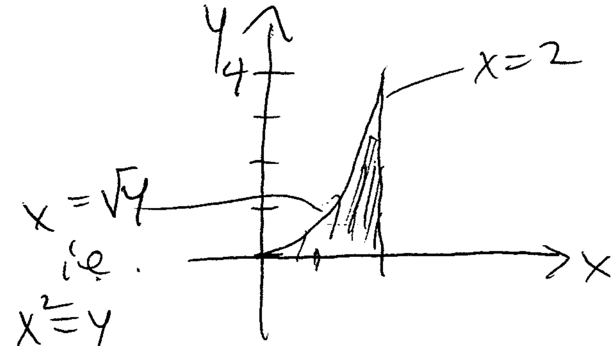
$$\text{volume} = \int_{r=0}^{r=2} \int_{\theta=0}^{\theta=2\pi} \int_{x=0}^{x=r\cos\theta+2} r\,dx\,d\theta\,dr$$

$$= \int_{r=0}^{r=2} \int_{\theta=0}^{\theta=2\pi} r(r\cos\theta+2)\,d\theta\,dr = \int_{r=0}^{r=2} \left(r^2\sin\theta + 2r\theta\right)\Big|_{\theta=0}^{\theta=2\pi} dr$$

$$= \int_{r=0}^{r=2} (2\pi\cdot 2r)\,dr = 2\pi r^2\Big|_0^2 = \boxed{8\pi}$$

Problem 4

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{y e^{x^2}}{x^3}\, dx\, dy$$



So, region may also be described by

$$\left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le x \le 2, \\ 0 \le y \le x^2 \end{array} \right\}.$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} \frac{y e^{x^2}}{x^3}\, dy\, dx$$

$$= \int_{x=0}^{x=2} \frac{y^2}{2} \frac{e^{x^2}}{x^3} \Big|_{y=0}^{y=x^2} dx = \int_{x=0}^{x=2} \frac{x^4}{2} \cdot \frac{1}{x^3} e^{x^2}\, dx$$

$$= \frac{1}{2} \int_0^2 x e^{x^2}\, dx = \frac{1}{4} \int_0^2 2x e^{x^2}\, dx = \frac{1}{4} e^{x^2} \Big|_0^2$$

$$= \frac{1}{4}\left(e^4 - e^0\right) = \boxed{\frac{1}{4}\left(e^4 - 1\right)}$$

Problem 5

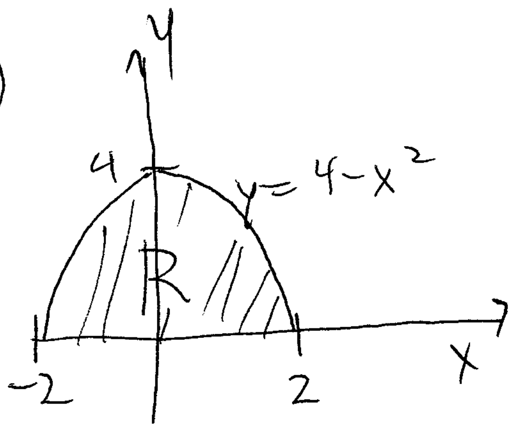
(1)

$$\text{mass} = \iint_R \delta(x,y)\,dx\,dy$$

centroid $= (\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{\text{mass}} \iint_R x\,\delta(x,y)\,dx\,dy, \qquad \bar{y} = \frac{1}{\text{mass}} \iint_R y\,\delta(x,y)\,dx\,dy.$$

(2)



$$\text{mass} = \iint_R 1\,dx\,dy$$

$$R = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le y \le 4 - x^2 \\ -2 \le x \le 2 \end{array} \right\}.$$

$$\text{mass} = \int_{-2}^{2} \int_{0}^{4-x^2} dy\,dx = \int_{-2}^{2} (4 - x^2)\,dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 - \left(-\frac{8}{3}\right)\right) = 16 - \frac{16}{3} = \boxed{\frac{32}{3}}.$$

Centroid region is symmetric — unchanged by reflection across y-axis — so $\bar{x} = 0$.

$$\bar{y} = \frac{1}{\text{mass}}\int_{-2}^{2}\int_{0}^{4-x^2} y\,dy\,dx = \frac{1}{m}\int_{-2}^{2} \frac{y^2}{2}\Big|_{y=0}^{y=4-x^2} dx$$

m = mass

$$= \frac{1}{m}\int_{-2}^{2} \frac{(4-x^2)^2}{2}\,dx = \frac{1}{2m}\int_{-2}^{2}\left(16 - 8x^2 + x^4\right)dx$$

$$= \frac{1}{2m}\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Big|_{-2}^{2}$$

$$= \frac{1}{2m}\left(32 - \frac{64}{3} + \frac{32}{5}\right) - \frac{1}{2m}\left(-32 + \frac{64}{3} - \frac{32}{5}\right)$$

$$= \frac{1}{2m}\left(64 - \frac{128}{3} + \frac{64}{5}\right) = \frac{1}{2m}\left(\frac{64\cdot 15 - 128\cdot 5 + 64\cdot 3}{15}\right)$$

$$= \frac{3}{64}\cdot\frac{512}{15} = \frac{3\cdot 64\cdot 8}{3\cdot 64\cdot 5} = \boxed{\frac{8}{5}}$$

So,

$$\boxed{\text{centroid} = \left(0, \frac{8}{5}\right)}$$

Problem 6

(1) $$I_0 = \iint_R (x^2+y^2)\,\delta(x,y)\,dx\,dy$$

(2). In polar coordinates,

$$R = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array} \right\}.$$

Get

$$I_0 = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r^2 \underbrace{(r\cos\theta)^2}_{\delta = x^2} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r^5 \cos^2\theta\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi} \cos^2\theta \cdot \frac{r^6}{6}\Big|_0^2\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \cdot \frac{64}{6}\,d\theta. \quad \text{Use} \quad \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta).$$

$$= \frac{64}{6}\int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(2\theta)\,d\theta = \frac{64}{6}\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]\Bigg|_0^{2\pi}$$

$$= \frac{64\pi}{6} = \boxed{\frac{32\pi}{3}}$$

Problem 7

(1) $u = y + 2x^2$, $v = y - 2x^2$.

$u + v = 2y$, so $y = \frac{1}{2}(u+v)$.

$u - v = y + 2x^2 - y + 2x^2 = 4x^2$, so

$\frac{u-v}{4} = x^2$ or $x = \frac{1}{2}\sqrt{u-v}$.

$$T(u,v) = \left(\frac{1}{2}\sqrt{u-v}, \frac{1}{2}(u+v)\right).$$

(2). $J_T = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ Now,

$\frac{\partial x}{\partial u} = \frac{1}{4}(u-v)^{-1/2}$, $\frac{\partial x}{\partial v} = \frac{1}{4}(u-v)^{-1/2}\cdot(-1)$.

$\frac{\partial y}{\partial u} = \frac{1}{2}$ $\frac{\partial y}{\partial v} = \frac{1}{2}$.

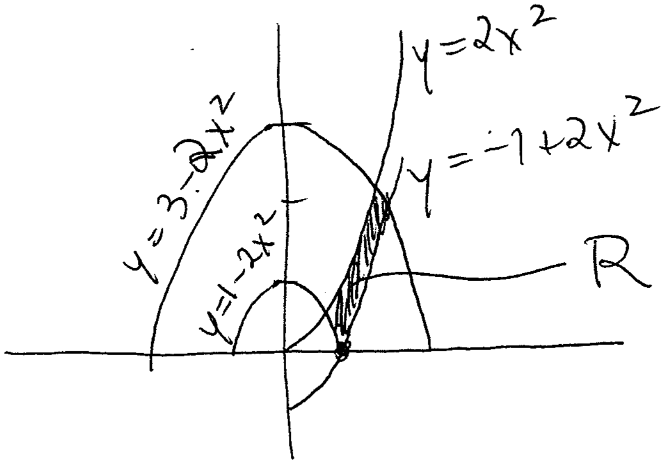
$$J_T = \begin{vmatrix} \frac{1}{4}(u-v)^{-1/2} & \frac{1}{4}(u-v)^{-1/2}(-1) \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{8}\frac{1}{\sqrt{u-v}} - -\frac{1}{8}\frac{1}{\sqrt{u-v}} = \frac{1}{4\sqrt{u-v}}.$$

(3). R is given by

$$\left.\begin{aligned} 1 \le y + 2x^2 \le 3, \\ -1 \le y - 2x^2 \le 0, \end{aligned}\right\} \quad \begin{aligned} 1 \le u \le 3 \\ -1 \le v \le 0. \end{aligned}$$

$x \geq 0.$



Now, using $x = \frac{1}{2}\sqrt{u-v}$, $y = \frac{1}{2}(u+v)$, get

$$x^2 + 2x + y^2 = \frac{1}{4}|u-v| + \sqrt{u-v} + \frac{1}{4}(u+v)^2.$$

$$= \frac{1}{4}(u-v) + \sqrt{u-v} + \frac{1}{4}(u+v)^2$$ on R since $u \geq v$ always on R.

So

$$\iint_R (x^2 + 2x + y^2)\,dx\,dy = \int_{v=-1}^{v=0}\int_{u=1}^{u=3} \left[\frac{1}{4}(u-v) + \sqrt{u-v} + \frac{1}{4}(u+v)^2\right] \overbrace{\frac{1}{4\sqrt{u-v}}}^{|J_T|}\,du\,dv$$

<u>Note</u> No need to take absolute value of J_T, it is already positive in this example!