Another Way To Compute Determinants

Here's an alternative "yoga" for computing a 3x3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Recopy the first two columns over to the right:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{matrix}$$

Now, multiply diagonals "top left to lower right" and add:

$$\left|\begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}\right.$$

$$a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$

Now multiply diagonals "top right to lower left" and put minus signs in front (like 2x2 determinants):

$$\left|\begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}\right.$$

$$- a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3$$

Finally, add it all up:

$$a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$

$$+$$

$$- a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3$$

$$\boxed{\begin{aligned} = {} & a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 \\ & - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3 . \end{aligned}}$$

<u>Example</u> Compute $\begin{vmatrix} 1 & 0 & 2 \\ -7 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$.

$$\left|\begin{array}{ccc|cc} 1 & 0 & 2 & 1 & 0 \\ -7 & 5 & 0 & -7 & 5 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right.$$

$$= 1\cdot 5\cdot 3 + 0\cdot 0\cdot 0 + 2\cdot(-7)\cdot 0$$

$$- 2\cdot 5\cdot 0 - 1\cdot 0\cdot 0 - 0\cdot(-7)\cdot 3$$

$$= 15.$$

To keep ourselves honest, let's just check that this formula agrees with our previous formula:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1c_2b_3 - b_1a_2c_3 + c_1a_2b_3 + b_1c_2a_3 - c_1b_2a_3.$$

This matches the formula in the box above!