Math 241 Section BL1

Exam Prep Problems (CORRECTED, APRIL 28, 9:30 A.M.)

Problem 1. Let B_R denote the ball of radius R in 3-dimensional space:

$$B_R = \{(x, y, z) \mid x^2 + y^2 + z^2 \le R^2\}.$$

(a) Compute the volume of B_R as a function $V(R)$ of R using a triple integral. Show your work.

(b) Compute the surface area of the sphere

$$S_R = \{(x, y, z) \mid x^2 + y^2 + z^2 = R^2\}$$

as a function $A(R)$ of R using a surface integral.

(c) It is a fact (which should be obvious from your calculations) that $V'(R) = A(R)$. Explain briefly why.

Problem 2. Let $g(x, y) = (x - 2y^2)(x - 3y^2)$.

(a) Let $\mathbf{r}(t) = \langle at, bt \rangle$ be a parametrized line through the origin (where a, b are some constants, not both zero). Show that $g(\mathbf{r}(t))$ has a local minimum at $t = 0$ for every choice of a and b (note that the case $a = 0$ and b nonzero will require a separate argument from the case $a \neq 0$).

(b) Graph the level set $g(x, y) = 0$ in the plane. On the same graph, label the regions where $g(x, y)$ is positive (with a $+$) and where $g(x, y)$ is negative (with a $-$).

(c) Show that g has a critical point at $(0, 0)$.

(d) Compute the Hessian of g at the point $(0, 0)$. What does the Second Derivative Test tell you about the critical point $(0, 0)$?

(e) Is $(0, 0)$ a local maximum, local minimum, or neither for $g(x, y)$? Justify your answer.

Problem 3. (12 points) Let

$$x(u, v) = 2u^2, \qquad y(u, v) = 3v - 2u$$

be functions of the variables u and v. Suppose $f(x, y)$ is a function of the variables x and y and let

$$F(u, v) = f(x(u, v), y(u, v))$$

(the composite function). Suppose also that we have the following tables of values:

(x,y)	$\frac{\partial f}{\partial x}(x,y)$
$(1,0)$	-3
$(2,1)$	4
$(2,3)$	1
$(3,1)$	2
$(3,2)$	-7

(x,y)	$\frac{\partial f}{\partial y}(x,y)$
$(1,0)$	-2
$(2,1)$	-1
$(2,3)$	2
$(3,1)$	7
$(3,2)$	3

Compute

$$\frac{\partial F}{\partial u}(1,1).$$

Justify your answer.

Problem 4. Let

$$f(x,y) = y - 2x^2.$$

(a) Use the method of Lagrange multipliers to find the maximum and minimum of $f(x,y)$ subject to the constraint $16x^2 + y^2 = 64$. Show your work.

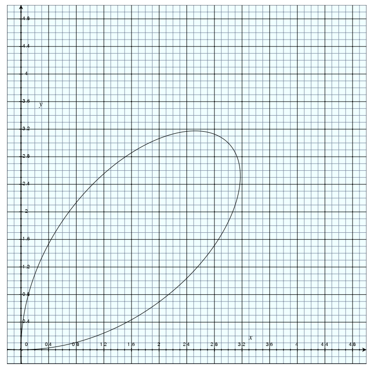
(b) Graph the ellipse $16x^2 + y^2 = 64$. Label your axes.

(c) List the values (a,b) for which $\nabla(f)(a,b)$ and $\nabla(g)(a,b)$ (usual notation) are parallel. For each such value of (a,b), let $c = f(a,b)$ and graph the level set $f(x,y) = c$ on the same set of axes as the ellipse. Show carefully how each parabola intersects the ellipse. **Explain why they should be tangent at the points of intersection!**

Problem 5. Find the volume of the solid lying inside the ellipsoid $4x^2 + 4y^2 + z^2 = 80$ and above the surface $z = 2x^2 + 2y^2$.

Problem 6. Let $\mathbf{r} = \langle x, y, z\rangle$. Let c be a constant. Let $\mathbf{F}(x,y,z) = \frac{c}{\|\mathbf{r}\|^3} \cdot \mathbf{r}$. Let S denote the sphere of radius 30 centered at the point $(1,2,0)$. Compute $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$. Explain what your method has to do with Gauss's Law (p. 1196 of the text).

Problem 7. Consider the region in the first quadrant of the plane enclosed by the curve C with equation $x^3 + y^3 = 6xy$:



(a) Find a parametrization $\mathbf{r}(t)$ of this curve.

(b) Compute the area of the region enclosed by the curve by computing the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y) = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle.$$

Problem 8. The average distance $\overline{D}$ from a fixed point P to a point (x, y) on a planar curve C is given by

$$\overline{D} = \frac{1}{s} \int_C D(x, y)\ ds$$

where $D(x, y)$ is the distance from P to (x, y) and s is the arclength of C.

Consider the spiral parametrized by $\vec{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t \rangle$ starting at $(1, 0)$ and approaching the origin as $t \to \infty$. Find the average distance from the origin to points on the spiral.

Problem 9. Consider the surface integral

$$\iint_\Sigma z\ dS$$

where Σ is the surface with sides S_1 given by the cylinder $x^2 + y^2 = 1$, S_2 given by the unit disk in the xy-plane, and S_3 given by the plane $z = x + 1$. Evaluate this integral as follows:

(a) Parametrize S_1 using (θ, z) coordinates.

(b) Evaluate the integral over the surface S_2 without parametrizing.

(c) Parametrize S_3 in (Des)cartesian coordinates and evaluate the resulting integral using polar coordinates.

(d) Combine the results to get $\iint_\Sigma z\ dS$.

Problem 10. Let P denote the parallelogram in space two of whose sides are the line segment from $(0,0,1)$ to $(1,2,3)$ and the line segment from $(0,0,1)$ to $(3,4,5)$.

(a) Without using an integral, compute the surface area of the parallelogram P.

(b) Find a parametrization $\mathbf{R}(x,y)$ of the surface P. Then, compute the surface area of P via a surface integral $\iint_P 1\,dS$ using the parametrization $\mathbf{R}(x,y)$.

Problem 11. Let S denote the "northern hemisphere" of the unit sphere, that is, the surface

$$S = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}.$$

Compute the flux of the vector field

$$\mathbf{F}(x,y,z) = \langle yz^2 + y, 2z - x^2, y + 1\rangle$$

[**A TYPO IN THIS VECTOR FIELD WAS CORRECTED, MONDAY, APRIL 28, 9:30 A.M.**] across S where S is given the upward-pointing unit normal vector, in two ways:

(a) Apply the Divergence Theorem. [Hint: the surface is not closed. Yet.]

(b) Apply Stokes's Theorem.

Problem 12. Suppose $h(x,y)$ is a function for which

$$f(1,2) = 3 \text{ and } \nabla f(1,2) = \langle 7, 11\rangle.$$

(a) Use linear approximation to estimate $f(1.2, 1.9)$.

(b) Suppose the Hessian of f at $(1,2)$ is:

$$H(f)(1,2) = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Is your estimate likely to be lower than the actual value, higher than the actual value, or is there not enough information to make a reasonable guess? Explain.