PS 22 Selected Answers/Partial Solns.

① πab

② Assuming constant density, symmetry about $x=0 \Rightarrow \bar{x}=0$. Now,

$$m = \rho\iint_R 1\,dA = \rho(1/2)6\pi$$
$$= 3\pi\rho$$

by Problem 1.

$$\bar{y} = \frac{1}{m}M_x = \left(\frac{1}{3\pi\rho}\right)\rho\iint_R y\,dA$$

(switch to elliptical coordinates) $$= \frac{1}{3\pi}\int_0^{\pi}\int_0^1 2r^2\sin\theta\,dr\,d\theta$$

$$= \left(\frac{1}{3\pi}\right)\left(4/3\right) = \frac{4}{9\pi}.$$

So, the center of mass is $\left(0, \frac{4}{9\pi}\right)$.

(3) Let $u = y/x$ and $v = xy$. Then,

$uv = y^2$ and $\frac{v}{u} = x^2$. So,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{-v}{2u^2\sqrt{v/u}} & \frac{1}{u\sqrt{v/u}} \\ \frac{v}{2\sqrt{uv}} & \frac{u}{\sqrt{uv}} \end{vmatrix} = \frac{-vu^{-1}}{2\sqrt{\left(\frac{v}{u}\right)}\sqrt{uv}}$$

$$= -\frac{1}{2u}$$

Note that $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2u}$ because $1 \le u \le 2$.

So, the area we want is

$$\frac{1}{2}\int_1^2\int_1^2 \frac{1}{u}\,du\,dv = \frac{1}{2}\ln(2).$$

(4) Let $u = x - y$ and $v = x + y$. So, $0 \le v \le 1$ and $-v \le u \le v$. Also,

$\frac{\partial(x,y)}{\partial(u,v)} = 1/2$. Then,

$$\iint_R \exp\left(\frac{x-y}{x+y}\right) dA = \frac{1}{2}\int_0^1\int_{-v}^{v} \exp\left(\frac{u}{v}\right) du\,dv$$

$$= \frac{e^2-1}{4e}.$$

(6) Let $u = xy$ and $v = x^2 - y^2$. Then,

$$\frac{\partial(u,v)}{\partial(x,y)} = -2(x^2+y^2). \quad (*)$$

Note that $(x^2+y^2)^2 = 4u^2 + v^2$, so that

$$(*) = -2\sqrt{4u^2+v^2}.$$

By the result of problem 5,

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2\sqrt{4u^2+v^2}}.$$

Hence,

$$\iint_R x^2+y^2 \, dA = \int_1^4\int_1^3 \frac{\sqrt{4u^2+v^2}}{2\sqrt{4u^2+v^2}} \, du \, dv$$

$$= 3.$$