Math 241 §BL1

Problem Set 33

(1) (a) Suppose $\vec{F}(x, y, z) = \langle P, Q, R\rangle$ is a continuously differentiable vector field on $\mathbb{R}^3$. Show that if $\vec{F}$ is conservative, then

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x},\\ \frac{\partial P}{\partial z} &= \frac{\partial R}{\partial x}, \text{ and}\\ \frac{\partial Q}{\partial z} &= \frac{\partial R}{\partial y}.\end{aligned}$$

(b) Let $\vec{F} = \langle y, x, xyz\rangle$. Show that the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

is not independent of path.

(2) Let $\vec{F}(x, y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y}\rangle$ and C be the curve that consists of the arc of the curve $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0, 0)$. Use Green's Theorem to compute

$$\int_C \vec{F} \cdot d\vec{r}.$$

(3) Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = \langle e^x + x^2y, e^y - xy^2\rangle$ and C is the circle of radius 5 centered at the origin oriented counterclockwise.

(4) Suppose a simple closed curve C bounds a simply conncted domain $\mathcal{D}$, and $\vec{F} = \langle P, Q\rangle$ is a continuously differentiable vector field on $\mathbb{R}^2$. Use Green's Theorem

to prove that $\vec{F}$ is conservative on $\mathcal{D}$ if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

(5) (a) Let C be the straight line segment in the plane from (x_1, y_1) to (x_2, y_2). Show that

$$\int_C \langle -y, x\rangle \cdot d\vec{r} = x_1 y_2 - x_2 y_1.$$

(b) Suppose that the vertices of a polygon are given by $(x_1, y_1), \ldots, (x_n, y_n)$ (the order is given by traversing the polygon in a counterclockwise fashion). Show that the area of the polygon is given by

$$\frac{1}{2}\sum_{j=1}^{n}(x_j y_{j+1} - x_{j+1} y_j)$$

where $x_{n+1} = x_1$ and $y_{n+1} = y_1$. (Hint: Use Green's Theorem).

(6) Suppose that f is a twice continuously differentiable function defined on a region R in the plane that is bounded by a smooth positively oriented simple closed curve C. Show

$$\oint_C \frac{\partial f}{\partial x}\ dy - \frac{\partial f}{\partial y}\ dx = \int\int_R \mathcal{L}(f)\ dx\ dy$$

where $\mathcal{L}(f)$ is the Laplacian of f. If a vector field is a gradient field of a harmonic function, what is the work done by this vector field in moving a particle along any simple closed curve in the plane? (Hint: see Problem Set 9 for definitions of words you may not remember).