Math 241 §BL1

Problem Set 9

(1) Determine all real numbers λ such that a function $f(x, y)$ having continuous second order partial derivatives has first order partials that satisfy

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= x + 2y^2 \\ \frac{\partial f}{\partial y}(x, y) &= \lambda^2 xy - y^3. \end{aligned}$$

What is f (for a correct choice of λ, of course)?

(2) The **Laplacian** $\Delta f(x, y)$ of a function $f(x, y)$ is defined to be

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y).$$

A function is said to be **harmonic** if $\Delta f \equiv 0$. Which of the following functions are harmonic?

(a) $f(x, y) = \ln \sqrt{x^2 + y^2}$

(b) $f(x, y) = e^{-x} \sin y$

(c) $f(x, y) = \sin x \sinh(\pi - y)$

(3) Let $f(x, y) = \sqrt{4 - x^2 - y^2}$. Define a vector-valued function $\vec{G}(x, y)$ by

$$\vec{G}(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right\rangle.$$

Show that $\vec{G}(x, y)$ has length equal to $\dfrac{2}{f(x, y)}$.

(4) Let

$$f(x, y) = \int_0^y \sqrt{x + t}\, dt$$

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(5) Suppose x and y are given as functions of variables r and θ as follows:

$$\begin{aligned} x(r,\theta) &= r\cos\theta \\ y(r,\theta) &= r\sin\theta. \end{aligned}$$

Consider the matrix

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

Compute the determinant of J.