STABILITY AND HAMILTONIAN REDUCTION FOR GROTHENDIECK-SPRINGER RESOLUTIONS

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ABSTRACT. Let $G = GL_n(\mathbb{C})$ and $\widetilde{\mathfrak{gl}}_n$ the Grothendieck-Springer resolution of its Lie algebra. Assuming that the zero pre-image $\mu^{-1}(0)$ of the moment map $\mu : T^*(\widetilde{\mathfrak{gl}}_n \times \mathbb{C}^n) \to \mathfrak{gl}_n^*$ is a complete intersection, we compute its irreducible components. These components dominate components of the corresponding moment pre-image for $T^*(\mathfrak{gl}_n \times \mathbb{C}^n)$. We then analyze GIT stability of the irreducible components of $\mu^{-1}(0)$ for various stability conditions. Unlike the case of $T^*(\mathfrak{gl}_n \times \mathbb{C}^n)$, in which GIT quotients—both isomorphic to the Hilbert scheme $(\mathbb{C}^2)^{[n]}$—can arise from only two of the $n+1$ irreducible components, every component of $\mu^{-1}(0)$ appears as a GIT quotient of $\mu^{-1}(0)$ in the Grothendieck-Springer setting.

1. Introduction

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, $V = \mathbb{C}^n$, and $G = GL_n(\mathbb{C})$. The cotangent bundle of the quotient stack $(\mathfrak{g} \times V)/G$ has a concrete description via Hamiltonian reduction. Namely, let $\mu : \mathfrak{g} \times \mathfrak{g} \times V \times V^* \to \mathfrak{g}$ be the map defined by $\mu(X, Y, i, j) = [X, Y] + ij$. Then $T^*((\mathfrak{g} \times V)/G) = \mu^{-1}(0)/G$.

This construction appears in a number of different places. If one restricts to the open set $\mu^{-1}(0)^s$ of points that are det-stable, that is, stable for the twist of the G-action by the determinant character, then the quotient $\mu^{-1}(0)^s/G$ is isomorphic to the Hilbert scheme $(\mathbb{C}^2)^{[n]}$. If one deforms to the twisted cotangent bundle $\mu^{-1}(-\hbar I)/G$ for nonzero $\hbar$, one gets the rational Calogero-Moser phase space. If one quantizes the ring of functions on the cotangent bundle to differential operators, one gets the rational Cherednik algebras of Etingof-Ginzburg [EG].

In nature, it is the open set of the full cotangent bundle, the Hilbert scheme, that plays the dominant role. For example, the Calogero-Moser space is a hyperkähler rotation of the Hilbert scheme [W]; and the Cherednik algebras naturally microlocalize over the Hilbert scheme [GS1, GS2, KR]. Moreover, although the full cotangent bundle $T^*((\mathfrak{g}\times V)/G)$ is a complete intersection that has $n+1$ irreducible components [GG], one only sees two of the components via classical geometry: one of them is the component in which the Hilbert scheme, the stable locus for det, is dense and the other is the component containing the stable locus for det^{-1}.

Because the whole of $T^*((\mathfrak{g} \times V)/G)$ has a nice interpretation as a moduli stack of perverse coherent sheaves [BN], however, it seems reasonable to wonder whether the other irreducible components can be singled out by, for example, appropriate Bridgeland stability conditions. This seems nontrivial,[1] however, and in this note we do something much easier. We replace $\mathfrak{g}$ by the partial Grothendieck-Springer

[1] I've been told that, according to experts, it should not be possible using "limit stability" as in [Ba, To].

resolution $\widetilde{\mathfrak{g}} = G \times_B \mathfrak{b} \to \mathfrak{g}$. We then prove that every irreducible component of $T^*((\mathfrak{g} \times V)/G)$ is dominated by at least one component of $T^*((\widetilde{\mathfrak{g}} \times V)/G)$. More precisely:

Theorem. *Suppose that the cotangent bundle $T^*((\widetilde{\mathfrak{g}} \times V)/G)$ is a complete intersection. Then the rational map from $T^*((\widetilde{\mathfrak{g}} \times V)/G)$ to $T^*((\mathfrak{g} \times V)/G)$ induced by the generically finite map $(\widetilde{\mathfrak{g}} \times V)/G \to (\mathfrak{g} \times V)/G$ defines a surjective map from the set of irreducible components of $T^*((\widetilde{\mathfrak{g}} \times V)/G)$ to the set of irreducible components of $T^*((\mathfrak{g} \times V)/G)$.*

We then analyze a collection of GIT stability conditions, and prove that on $T^*((\widetilde{\mathfrak{g}} \times V)/G)$ there are enough GIT stability conditions to pick out each component separately.

Theorem. *Suppose that the cotangent bundle $T^*((\widetilde{\mathfrak{g}} \times V)/G)$ is a complete intersection. Then for every irreducible component $\mathcal{C}$ of $T^*((\widetilde{\mathfrak{g}} \times V)/G)$, there is a line bundle $\mathcal{L}$ on $T^*((\widetilde{\mathfrak{g}} \times V)/G)$ with the properties:*

(1) (Proposition 5.4) *The G-semistable locus with respect to $\mathcal{L}$ has empty intersection with every irreducible component $\mathcal{C}' \neq \mathcal{C}$.*
(2) (Proposition 5.5) *The G-semistable locus in $\mathcal{C}$ is nonempty, hence open and dense in $\mathcal{C}$.*

Analogous results (with identical proofs) hold when $(\mathfrak{g} \times V)/G$ is replaced by the perverse symmetric product of any smooth curve [BN]; this requires more terminology but no new ideas or calculations, so we leave this as an exercise for the interested reader.

It would be interesting to know whether the different GIT quotients, which (under the complete intersection hypothesis) are all birational and symplectic, are in fact derived equivalent; and furthermore whether there's any nice connection between one (or more) of these GIT quotients and the isospectral Hilbert scheme [H] (given that both are birational to the same variety).

2. Background

Let $X \xrightarrow{f} Y$ be a smooth morphism of smooth varieties. We get a closed immersion, "pullback of forms," $T^*Y \times_Y X \xrightarrow{i} T^*X$.

Suppose G is an algebraic group acting compatibly on X and Y. We get moment maps

$$T^*X \xrightarrow{\mu_X} \mathfrak{g}^*, \;\; T^*Y \xrightarrow{\mu_Y} \mathfrak{g}^*.$$

Note that the quotient stack $\mu_X^{-1}(0)/G$ is (essentially by definition) the cotangent bundle $T^*(X/G)$ of the quotient stack X/G (it is perhaps a bit misleading to call it a bundle, since it is generally not smooth, though it is often a local complete intersection).

Lemma 2.1. *The diagram*

$$\begin{array}{ccc} T^*Y \times_Y X & \xrightarrow{\mu_X \circ i} & \mathfrak{g}^* \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \mu_Y} & \\ T^*Y & & \end{array}$$

commutes.

Now, suppose $X \xrightarrow{f} Y$ is a principal P-bundle for a group P.

Lemma 2.2. *The image of $T^*Y \times_Y X \xrightarrow{i} T^*X$ is the kernel of the moment map $\mu^P : T^*X \to \mathfrak{p}^*$.*

Corollary 2.3.

(1) *Suppose $X \xrightarrow{f} Y$ is a principal P-bundle and X, Y are equipped with compatible G-actions commuting with the P-action. Then $\mu_Y^{-1}(0) \times_Y X = (\mu^{G\times P})^{-1}(0)$, where $\mu^{G\times P} : T^*X \to \mathfrak{g}^* \times \mathfrak{p}^*$ is the moment map for the $G \times P$-action.*
(2) *In this case, $\mu_Y^{-1}(0)/G \cong (\mu^{G\times P})^{-1}(0)/G \times P$.*

3. Cotangent Bundle of Grothendieck-Springer Resolution

Let G be a complex reductive group. Let $\mathfrak{g} = \mathrm{Lie}(G)$ with the adjoint action, and let V be a representation of G. Let P be a parabolic subgroup of G. Let $Y = \widetilde{\mathfrak{g}} \times V$, where $\widetilde{\mathfrak{g}} = G \times_P \mathfrak{p}$ is the partial Grothendieck-Springer resolution of $\mathfrak{g}$. Then $X = G \times \mathfrak{p} \times V$ is a principal P-bundle over Y. In addition, $T^*X = G \times \mathfrak{g}^* \times \mathfrak{p} \times \mathfrak{p}^* \times V \times V^*$, and the moment map $\mu_X : T^*X \to \mathfrak{g}^*$ is given by $\mu_X(g, \theta, r, s, i, j) = \theta + a^*(ij)$ where $a : \mathfrak{g} \to \mathrm{End}(V)$ is the infinitesimal action.

Moreover, the moment map $\mu_P : G \times \mathfrak{g}^* \times \mathfrak{p} \times \mathfrak{p}^* \times V \times V^* \to \mathfrak{p}^*$ for the P-action is given by $\mu^P(g, \theta, r, s, i, j) = \overline{\mathrm{ad}_g^*(\theta)} + \mathrm{ad}_r^*(s)$ where $\overline{v}$ means the image of $v \in \mathfrak{g}^*$ under the projection $\mathfrak{g}^* \to \mathfrak{p}^*$. Indeed, consider P acting on G via $b \cdot g = gb^{-1}$.

Claim 3.1. The moment map $\phi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ for the action of G on T^*G induced by $g \cdot g' = g'g^{-1}$ on G is $\phi(g, v) = \mathrm{Ad}_g(v)$.

Given Claim 3.1, we are just restricting the action to P, hence the formula. The proof of Claim 3.1 just depends on the following: $g \in G$ acts on TG by $g \cdot (g', Z) = (g'g^{-1}, \mathrm{Ad}_g(Z))$.

We draw the following conclusion for the $G \times P$-action. We have:

$$(\mu^{G\times P})^{-1}(0) = \big\{(g, \theta, r, s, i, j) \bigm| \overline{\mathrm{ad}_g^*(\theta)} = -\mathrm{ad}_r^*(s), \theta = -a^*(ij)\big\}.$$

Making this more concrete for $G = GL_n$, P a "standard" parabolic (i.e. consisting of block upper triangular matrices) and $V = \mathbf{C}^n$, so a is the identity map, we get:

$$(\mu^{G\times P})^{-1}(0) = \big\{(g, \theta, r, s, i, j) \bigm| \dagger \quad \text{holds}\big\},$$

where

$$(\dagger) \leftrightarrow \begin{cases} \theta = -ij, \\ \overline{g\theta g^{-1}} = -ad_r^*(s). \end{cases}$$

Now, note that G acts freely on $(\mu^{G\times P})^{-1}(0)$—the equations ($\dagger$) should be understood as constraints on θ and r, s, not on $g \in G$—and we may use the G-action to make $g = 1$, the identity element of G. Let $\mu^P : \mathfrak{p} \times \mathfrak{p}^* \times \mathbf{C}^n \times (\mathbf{C}^n)^* \to \mathfrak{p}^*$ be the moment map for the P-action, defined by $\mu^P(r, s, i, j) = \mathrm{ad}_r^*(s) + \overline{ij}$. We then get:

Proposition 3.2. *The natural inclusion $(\mu^P)^{-1}(0) \hookrightarrow (\mu^{G\times P})^{-1}(0)$ defined by $(r, s, i, j) \mapsto (1, -ij, r, s, i, j)$ induces a bijection between the set of P-orbits on $(\mu^P)^{-1}(0)$ and the set of $G \times P$-orbits on $(\mu^{G\times P})^{-1}(0)$. This gives an isomorphism of quotient stacks $(\mu^P)^{-1}(0)/P \cong (\mu^{G\times P})^{-1}(0)/G \times P$.*

Corollary 3.3. *There is an isomorphism of quotient stacks*

$$(\mu^P)^{-1}(0)/P \cong T^*(\tilde{\mathfrak{g}} \times \mathbf{C}^n/GL_n).$$

In light of Corollary 3.3, to study the Hamiltonian reduction of $T^*(\widetilde{\mathfrak{g}} \times V)$ under the G-action, we may work with the moment map μ^P instead.

Now let $\pi : T^*(\mathfrak{p} \times V) \to \mathfrak{p}/\mathfrak{n}$ denote the projection $(X, Y, i, j) \mapsto X + \mathfrak{n} \in \mathfrak{p}/\mathfrak{n}$, where $\mathfrak{n}$ denotes the nilradical of $\mathfrak{p}$. Specializing to the case $P = B$ of a Borel, I conjecture:

Conjecture 3.4. The map $\mu^B \times \pi$ is flat.

The conjecture has as a corollary:

Corollary 3.5. *Suppose Conjecture 3.4 holds. Then the fiber $\mu^{-1}(0)$ is a complete intersection in $T^*(\mathfrak{b} \times V)$ of dimension $2\dim(\mathfrak{b} \times V) - \dim(B)$.*

Proof. First, note that $\mu(0, Y, 0, 0) = 0$, so $(0, Y, 0, 0) \in \pi^{-1}(0)$, i.e. $\pi|_{\mu^{-1}(0)}$ is surjective. Base-changing the flat morphism $\mu \times \pi$ along the morphism $\{0\} \times \mathfrak{b}/\mathfrak{n} \to \mathfrak{b}^* \times \mathfrak{b}/\mathfrak{n}$, which is a complete intersection, we find that $\mu^{-1}(0) \hookrightarrow T^*(\mathfrak{b} \times V)$ is a complete intersection and $\pi|_{\mu^{-1}(0)}$ is flat. □

I have checked the corollary for small ($n \leq 4$) values of n using Macaulay 2. This corollary has apparently recently been proven by Mee Seong Im.

4. Description of Irreducible Components

For the remainder of the paper, we specialize to the case $P = B$.

Suppose $X = \mathrm{diag}(x_1, \ldots, x_n)$ is a diagonal $n \times n$ matrix. Then the (i, j)th entry $[X, Y]_{ij}$ of $[X, Y]$ for any matrix Y is $(x_i - x_j)y_{ij}$. If $x_i \neq x_j$ for all $i \neq j$, we get $([X, Y] + IJ)_{ab} = (x_a - x_b)y_{ab} + (IJ)_{ab}$ for any $I \in \mathbf{C}^n, J \in (\mathbf{C}^n)^*$. Setting this equal to zero, we find

$$y_{ab} = \frac{-(IJ)_{ab}}{x_a - x_b} \tag{4.1}$$

for $a \neq b$. In particular, if the image of $[X, Y] + IJ$ is zero in $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{b}^*$, this equation shows that, provided X has distinct eigenvalues, we can solve for y_{ab} (for $a \neq b$) uniquely. Summarizing:

Lemma 4.1. *Suppose X is a diagonal $n \times n$ matrix with distinct eigenvalues, and fix $I \in \mathbf{C}^n$ and $J \in (\mathbf{C}^n)^*$. Then, given diagonal entries y_{aa} for an $n \times n$ matrix Y, there is a unique Y satisfying $[X, Y] + IJ = 0$. In particular, if $\overline{Y} \in \mathfrak{g}/\mathfrak{n}$ and $\mu^P(X, Y, I, J) = 0$, there is a unique lift of $\overline{Y}$ to $Y \in \mathfrak{g}$ such that $\mu^G(X, Y, I, J) = 0$.*

We thus get the following description of irreducible components of $(\mu^B)^{-1}(0)$:

Proposition 4.2. *Assume Conjecture 3.4. The irreducible components of $(\mu^B)^{-1}(0)$ are the closures of subsets C_ℓ, where $\ell : \{1, \ldots, n\} \to \{0, 1\}$ is a function. The subset C_ℓ consists of the orbits of quadruples $(X, \overline{Y}, I, J)$ where X is diagonal, $I_k = \ell(k)$, $J_k = 1 - \ell(k)$, and $\overline{Y}$ is the image of a matrix $Y \in \mathfrak{g}$ that has arbitrary diagonal entries and off-diagonal entries determined by Equation (4.1).*

Proof. Consider first the subset $(\mu^B)^{-1}(0) \cap \pi^{-1}(\mathfrak{b}/\mathfrak{n} \smallsetminus \Delta)$, where $\Delta \subset \mathfrak{b}/\mathfrak{n}$ denotes the "big diagonal," the closed subset consisting of elements that have an eigenvalue with multiplicity at least two. By Lemma 4.1, every $(X, \overline{Y}, I, J) \in$

$(\mu^B)^{-1}(0) \cap \pi^{-1}(\mathfrak{b}/\mathfrak{n} \smallsetminus \Delta)$ lifts to a quadruple $(X, Y, I, J) \in (\mu^G)^{-1}(0)$. By the normal form for such quadruples from [GG], it follows that the sets C_ℓ as above are open sets in irreducible components of $(\mu^B)^{-1}(0) \cap \pi^{-1}(\mathfrak{b}/\mathfrak{n} \smallsetminus \Delta)$; hence their closures give components of $(\mu^B)^{-1}(0)$. Moreover, the conjecture implies that $\mu \times \pi$ is equidimensional, from which it follows that $\pi^{-1}(\Delta) \cap (\mu^B)^{-1}(0)$ has strictly smaller dimension than $(\mu^B)^{-1}(0)$, hence doesn't contribute irreducible components. □

5. Semistability for Components

We continue to specialize to the case $P = B$, a Borel subgroup, and write $\mu = \mu^B$. In this section, we analyze which irreducible components can contribute to the stable locus of $\mu^{-1}(0)$ under various choices of GIT stability conditions.

Recall first that a character $\phi : B \to \mathbf{C}^*$ defines a line bundle $L(\phi)$ on $(G \times_B \mathfrak{b}) \times V$ whose sections are defined by (B, ϕ)-semi-invariant functions on $(G \times \mathfrak{b}) \times V$: that is, functions $F : G \times \mathfrak{b} \times V \to \mathbf{C}$ for which $F(gb^{-1}, bX, v) = F(g, X, v)\phi(b)$. It follows that G-invariant sections of $L(\phi)$ are defined by functions $F : \mathfrak{b} \times V \to \mathbf{C}$ for which $F(bXb^{-1}, b \cdot v) = \phi(b)F(X, v)$ for $b \in B$. Similarly, pulling $L(\phi)$ back to $T^*((\widetilde{\mathfrak{g}} \times V)/G)$ and applying Corollary 3.3, we find:

Lemma 5.1. *The semistable locus $T^*((\widetilde{\mathfrak{g}} \times V)/G)^{ss}$ with respect to $L(\phi)$ is identified, via Corollary 3.3, with $\mu^{-1}(0)^{ss}/B$, where*

$$\mu^{-1}(0)^{ss} = \{(X, Y, I, J) \in \mu^{-1}(0) \subset T^*(\mathfrak{b} \times V) \mid (*) \text{ holds}\}.$$

Here we say $(*)$ holds for (X, Y, I, J) if there exists a function $F : \mu^{-1}(0) \to \mathbf{C}$ such that $F(X, Y, I, J) \neq 0$ and $F(b \cdot (X, Y, I, J)) = \phi(b)F(X, Y, I, J)$ for all $b \in B$.

Remark 5.2. The usual Hilbert-Mumford criterion, adapted to this setting as in [K, Lemma 2.4], tells us that a point (X, Y, I, J) fails to be ϕ-semistable if there exists a one-parameter subgroup $\lambda : \mathbf{C}^* \to B$ such that $\lim_{z \to 0} \lambda(z) \cdot (X, Y, I, J)$ exists and $\lim_{z \to 0} \phi^{-1}(\lambda(z)) = 0$.

We continue with some notation. Fix a function $\delta : \{1, 2, \ldots, n\} \to \{0, 1\}$. Let $\Delta = \delta^{-1}(1)$. Define

$$I = \begin{pmatrix} i_1 \\ \vdots \\ i_n \end{pmatrix} \text{ by } i_k = \delta(k) \text{ and } J = (j_1 \ldots j_n) \text{ by } j_k = 1 - \delta(k). \tag{5.1}$$

Suppose that $X = \mathrm{diag}(x_1, \ldots, x_n)$ is a diagonal matrix with $x_i \neq x_j$ for $i \neq j$. Let Y be an element of $\mathfrak{b}^*$ with diagonal entries $y_1, \ldots, y_n$ and

$$y_{k\ell} = \frac{\chi_\Delta(k)\chi_{\Delta^c}(\ell)}{x_k - x_\ell} \tag{5.2}$$

for $k > \ell$; here χ_Δ denotes the characteristic function of Δ and χ_{Δ^c} denotes the characteristic function of the complement of Δ. Defining Y this way, we get a quadruple $(X, Y, I, J) \in \mu^{-1}(0)$ with I, J as defined in the previous paragraph. By the results of the previous section, a generic quadruple in the component of $\mu^{-1}(0)$ labelled by Δ has a quadruple (X, Y, I, J) of the above form in its B-orbit.

Now define a character $\phi : B \to \mathbf{C}^*$ as follows: if $b \in B$ has diagonal entries $t_1, \ldots, t_n$, set $\phi^{-1}(b) = \prod_{k=1}^{n} t_k^{d_k}$, where $d_k = (-1)^{\delta(k)}$. For any cocharacter $\lambda(z) =$

$\mathrm{diag}(z^{a_1}, z^{a_2}, \dots, z^{a_n})$, we get

$$\phi^{-1}(\lambda(z)) = z^a \text{ where } a = \sum_{k=1}^{n} a_k d_k = \sum_{k=1}^{n} (-1)^{\delta(k)} a_k. \tag{5.3}$$

In particular, note:

Remark 5.3. Expression (5.3) is nonpositive provided both $a_k \geq 0$ whenever $k \in \Delta$ and $a_k \leq 0$ whenever $k \notin \Delta$.

Suppose now that $\delta' : \{1, \dots, n\} \to \{0, 1\}$ is some other choice of function and $\Delta' = (\delta')^{-1}(0)$. Suppose (X, Y, I, J) is a quadruple defined as above (with X diagonal, etc.). Then we have $\lambda(z)X\lambda(z)^{-1} = X$ and $(\lambda(z)Y\lambda(z)^{-1})_{k\ell} = z^{a_k - a_\ell} y_{kl}$. Since $y_{k\ell}$ is nonzero for $k \neq \ell$ if and only if $k \in \Delta', \ell \notin \Delta'$, we conclude that $\lim_{z\to 0} \lambda(z)Y\lambda(z)^{-1}$ will exist if and only if:

(a) The a_k satisfy $a_k - a_\ell \geq 0$ whenever $k \in \Delta'$, $\ell \notin \Delta'$, $k \neq \ell$.

Furthermore, $(\lambda(z) \cdot I)_k = z^{a_k} i_k$ and $(\lambda(z) \cdot J)_k = z^{-a_k}(1 - \delta'(k))$. These have limits as $z \to 0$ if and only if:

(b) The a_k satisfy $a_k \geq 0$ whenever $k \in \Delta'$ and $a_k \leq 0$ whenever $k \notin \Delta'$.

Note that if condition (b) is satisfied then so is condition (a).

Recall that we have defined a character $\phi : B \to \mathbf{C}^*$ above.

Proposition 5.4. *The ϕ-semistable locus of $\mu^{-1}(0)$ has empty intersection with the components of $\mu^{-1}(0)$ labelled by Δ' for $\Delta' \neq \Delta$. In particular, the ϕ-semistable locus is a subset of the Δ-component.*

Proof. Suppose that $\Delta' \neq \Delta$.

Suppose first that there exists $k_0 \in \Delta \setminus \Delta'$. Define a cocharacter λ as above by setting

$$a_k = \begin{cases} -1 & \text{if } k = k_0, \\ 0 & \text{otherwise.} \end{cases}$$

This choice satisfies the analogs of conditions (a) and (b) above for Δ', so $\lim_{z\to 0} \lambda(z) \cdot (X, Y, I, J)$ exists for the special representatives (X, Y, I, J) of generic points in the Δ'-component that we constructed above. On the other hand, using (5.3) we get $a = (-1)^{\delta(k_0)} a_{k_0} = (-1)(-1) = 1$, so that $\phi^{-1}(\lambda(z)) = z$. We conclude that the cocharacter λ we have chosen de(semi)stabilizes the generic point (X, Y, I, J) of the Δ' component by Remark 5.2.

Suppose next that there exists $k_1 \in \Delta \setminus \Delta'$. Define a cocharacter λ by setting

$$a_k = \begin{cases} 1 & \text{if } k = k_1, \\ 0 & \text{otherwise.} \end{cases}$$

This choice satisfies the analogs of (a) and (b) again, and $a = (-1)^{\delta}(k_1) a_{k_1} = (-1)^0 \cdot 1 = 1$ again. Completing the argument as above, we find that this choice of λ destabilizes the generic point of the Δ' component by Remark 5.2. $\square$

We also have the following:

Proposition 5.5. *With the choice of ϕ above, the generic point of the Δ-component of $\mu^{-1}(0)$ is ϕ-semistable.*

Proof. We will construct a ϕ-semi-invariant function F.

Let $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$ denote the standard increasing ($\mathfrak{b}$-invariant) filtration on V, and similarly let $V_i^* = V_i^\perp$: writing elements of V^* as n-dimensional row vectors, V_i^* consists of vectors whose left-most i entries are zero; in particular, it gives a decreasing filtration of V^*. For a matrix $X \in \mathfrak{b}$, let $\alpha_i(X)$ denote the ith diagonal entry (i.e. ith eigenvalue) of X.

Remark 5.6. Note that $X - \alpha_i(X)$ acts by zero on V_i/V_{i-1} and on V_{i-1}^*/V_i^*. In particular, $(X - \alpha_i(X)) \cdot V_i \subset V_{i-1}$ and $V_{i-1}^* \cdot (X - \alpha_i(X)) \subset V_i^*$.

Given i, let $k_i(X) = \prod_{j \neq i}(X - \alpha_j(X))$. Then, by an induction using Remark 5.6, we have $k_i(X) \cdot V \subset V_i$ and similarly $V^* \cdot k_i(X) \subset V_{i-1}^*$. Let $\pi_i : V_i \to V_i/V_{i-1}$ and $\pi_i^* : V_{i-1}^* \to V_{i-1}^*/V_i^*$ denote the projections (note the shift in indexing on the V^* side). Given $(X, Y, I, J) \in \mu^{-1}(0)$, define

$$\Pi_i(X, Y, I, J) = \pi_i(k_i(X)I) \in V_i/V_{i-1}, \qquad \Pi_i^*(X, Y, I, J) = \pi_i^*(Jk_i(X)) \in V_{i-1}^*/V_i^*.$$

Now, given $\delta : \{1, 2, \ldots, n\} \to \{0, 1\}$ and preserving earlier notation, define

$$\begin{aligned} F(X, Y, I, J) = &\left(\bigotimes_{k \in \Delta} \Pi_k(X, Y, I, J)\right) \otimes \left(\bigotimes_{k \in \Delta^c} \Pi_k^*(X, Y, I, J)\right) \\ &\in \left(\bigotimes_{k \in \Delta}(V_k/V_{k-1})\right) \otimes \left(\bigotimes_{k \in \Delta^c}(V_{k-1}^*/V_k^*)\right). \end{aligned}$$

If $b \in B$, note that

$$\Pi_i(b \cdot (X, Y, I, J)) = \pi_i(k_i(bXb^{-1})bI) = \pi_i(b \cdot k_i(X)I) = \alpha_i(b)\Pi_i(X, Y, I, J)$$

and similarly

$$\Pi_i^*(b\cdot(X, Y, I, J)) = \pi_i^*(Jb^{-1}k_i(bXb^{-1})) = \pi_i^*(Jk_i(X)b^{-1}) = \alpha_i(b)^{-1}\Pi_i^*(X, Y, I, J).$$

It is immediate that

$$F(b \cdot (X, Y, I, J)) = \left(\prod_{k \in \Delta} \alpha_k(b) \times \prod_{k \in \Delta^c} \alpha_k(b)^{-1}\right) F(X, Y, I, J) = \phi(b)F(X, Y, I, J).$$

In particular, F is ϕ-semi-invariant.

We now let $X = \mathrm{diag}(x_1, \ldots, x_n)$ be diagonal with $x_i \neq x_j$ for $i \neq j$, and define I, J, and Y as in (5.1) and (5.2) to get a particular quadruple $(X, Y, I, J) \in \mu^{-1}(0)$. Using the images of standard basis elements as basis vectors in V_i/V_{i-1} and V_{i-1}^*/V_i^*, we make the identifications

$$\pi_\ell(k_\ell(X)I) = \left(\prod_{j \neq \ell}(x_\ell - x_j)\right) I_\ell, \quad \pi_\ell^*(Jk_\ell(X)) = \left(\prod_{j \neq \ell}(x_\ell - x_j)\right) J_\ell.$$

It follows that, for such a quadruple (X, Y, I, J), since $I_\ell = \delta(\ell)$ (which equals 1 when $\ell \in \Delta$) and $J_\ell = 1 - \delta(\ell)$ (which equals 1 when $\ell \in \Delta^c$), we get $F(X, Y, I, J) \neq 0$. So such points are ϕ-semistable. In particular, the ϕ-semistable locus in the component of $\mu^{-1}(0)$ labelled by Δ is nonempty, open and dense. □

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