DESCENT OF COHERENT SHEAVES AND COMPLEXES TO GEOMETRIC INVARIANT THEORY QUOTIENTS: DRAFT

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Abstract. Fix a quasi-projective scheme X over a field of characteristic zero that is equipped with an action of a reductive algebraic group G. Fix a polarization H of X that linearizes the G-action. We give necessary and sufficient conditions for a G-equivariant coherent sheaf on X to descend to the GIT quotient $X/\!\!/G$, or for a bounded-above complex of G-equivariant coherent sheaves on X to be G-equivariantly quasi-isomorphic to a bounded-above complex of G-equivariant coherent sheaves that descends to the GIT quotient $X/\!\!/G$. We use this to give a description of the derived category of $X/\!\!/G$ in terms of a category of G-equivariant sheaves on X.

1. Introduction

Varieties constructed using geometric invariant theory (or GIT) have become ubiquitous in algebraic geometry; for example, the moduli spaces $M_{g,n}$ of stable n-pointed curves (see [HM98]), or moduli spaces of semistable vector bundles or, more generally, semistable coherent sheaves (see [Mar96]) on a fixed projective variety can be constructed as GIT quotients of Hilbert schemes or Quot schemes. Many fundamental questions in the study of such GIT quotients concern the properties of certain natural vector bundles (or Chern classes of vector bundles) on them. As notable examples, one might mention the essential role played by the classes κ_i and λ_j in the study of the geometry of $M_{g,n}$, or the remarkable "Verlinde formula" (see [Bea95]) describing the spaces of sections of powers of the determinant line bundle on the moduli space of semistable vector bundles on a curve.

In most cases one constructs such vector bundles on moduli spaces using a descent technique; this proceeds as follows. One begins by identifying the relevant moduli space as a GIT quotient $X/\!\!/G$ of a variety X by a reductive group G; here by a GIT quotient we mean what is typically called a *good quotient*—see Definition 5.4. A coherent sheaf (or complex of coherent sheaves) $\mathcal{M}$ on the semistable locus X^{ss} is said to *descend to* $X/\!\!/G$ if there is a coherent sheaf (or complex of coherent sheaves) $\overline{\mathcal{M}}$ on $X/\!\!/G$ for which, letting

$$X^{ss} \xrightarrow{\pi} X/\!\!/G$$

denote the quotient map (see [MFK94] or [New78]), there is a G-equivariant isomorphism $\pi^*\overline{\mathcal{M}} \cong \mathcal{M}$. With this notion in hand, one constructs the desired vector bundle on $X/\!\!/G$ indirectly, by identifying a vector bundle on X^{ss} that one expects to be the pullback of the desired bundle on $X/\!\!/G$ and then checking that the bundle on X^{ss} does indeed descend to $X/\!\!/G$.

Date: December 7, 2001.

The following theorem, which may be found in [DN89] (where the authors of that paper attribute it to Kempf), gives a convenient characterization of the vector bundles on X^{ss} that descend to $X/\!\!/G$.

Theorem 1.1. (see [DN89]) *Suppose X is a quasiprojective scheme over a an algebraically closed field k of characteristic zero, and that G is a reductive algebraic group over k that acts on X with a fixed choice of linearization H. Let E be a G-vector bundle on X^{ss}. Then E descends to $X/\!\!/G$ if and only if for every closed point x of X^{ss} such that the orbit $G \cdot x$ is closed in X^{ss}, the stabilizer of x in G acts trivially on the fiber E_x of E at x.*

In this paper we extend the descent criterion of Theorem 1.1 in two directions. First, we give a criterion for an arbitrary G-equivariant coherent sheaf to descend to $X/\!\!/G$.

Theorem 1.2. *Suppose that X is a quasiprojective scheme over a field k of characteristic zero and that G is a reductive algebraic group over k that acts on X with a fixed choice of linearization H. Let $\mathcal{M}$ denote a G-equivariant coherent $\mathcal{O}_X$-module on X^{ss}. Then the following are equivalent:*

1. *$\mathcal{M}$ descends to $X/\!\!/G$.*
2. *For every point $x \in X^{ss}$, closed or not, the $\mathcal{O}_{X,x}$-modules $\mathcal{M} \otimes \mathcal{O}_{X,x}/\mathfrak{M}_x$ and $\mathrm{Tor}_1^{\mathcal{O}_X}\left(\mathcal{M}, (\mathcal{O}_{X,x}/\mathfrak{M}_x)\right)$ are generated by elements invariant under the isotropy subgroup G_x.*

If k is algebraically closed, these are equivalent also to

3. *For every closed point $x \in X^{ss}$ that lies in a G-orbit that is closed in X^{ss}, the $\mathcal{O}_{X,x}$-modules $\mathcal{M} \otimes \left(\mathcal{O}_X/\mathfrak{M}_x\right)$ and $\mathrm{Tor}_1^{\mathcal{O}_X}\left(\mathcal{M}, \left(\mathcal{O}_X/\mathfrak{M}_x\right)\right)$ are trivial representations of the isotropy group G_x.*

The proof of Theorem 1.2 is given, along with the proof of Theorem 1.1, in Section 2.

This criterion gives a general new method for constructing coherent sheaves on moduli spaces, clearing the way for a broader study of the sheaf theory of moduli spaces than has previously been possible. For example, one might hope eventually to combine this result with the characterization given by Lehn (see [Leh98]) of the cotangent sheaves of Quot schemes to obtain new information about the cotangent sheaves of singular moduli spaces of semistable sheaves, or, more speculatively, with a suitable generalization of Lehn's work to characterize the entire cotangent complexes of moduli spaces of semistable sheaves. To carry out the latter kind of project, one would need to understand descent of complexes of sheaves.

Generalizing Theorem 1.1 in another direction, then, we also study the descent of equivariant complexes of coherent sheaves to $X/\!\!/G$. Here the question is both more complicated and more subtle: for one thing, one must be careful about what one means by "descent of a complex" when the complex is not a complex of vector bundles. The right point of view is provided by the derived category formalism: one has a pullback functor π^* from the category of coherent sheaves on $X/\!\!/G$ to the category of coherent sheaves on X^{ss}, and if one wants to use this to study the relationships between the derived categories of $X/\!\!/G$ and X^{ss}, then, when pulling back a complex $\mathbf{E}$ that does not consist of vector bundles, one ought to replace $\pi^*\mathbf{E}$ by $\mathbf{L}\pi^*\mathbf{E}$, the left derived functor of π^* applied to $\mathbf{E}$. The image of this left derived functor consists of complexes of vector bundles on X^{ss}, and consequently it is primarily with such complexes that we will concern ourselves.

Theorem 1.3. *Suppose X is a quasiprojective scheme over a field k of characteristic zero. Suppose G is a reductive algebraic group over k that acts on X with a fixed choice of linearization H. Let*

$$\mathbf{E}: \quad \cdots \to E_n \xrightarrow{\phi_n} E_{n-1} \xrightarrow{\phi_{n-1}} \ldots \to E_2 \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0 \to 0$$

denote a bounded-above G-equivariant complex of vector bundles on X^{ss}. Then the following are equivalent:

1. $\mathbf{E}$ *is equivariantly quasi-isomorphic to a complex $\mathbf{E}'$ of vector bundles on X^{ss} that descends to $X/\!\!/G$.*
2. *For every point $x \in X^{ss}$, closed or not, the $\mathcal{O}_X$-modules*
$$\mathcal{H}^{-j}\big(\mathbf{E} \otimes (\mathcal{O}_{X,x}/\mathfrak{M}_x)\big)$$
are generated by elements invariant under the isotropy subgroup G_x for all j.

If k is algebraically closed, these are equivalent also to

3. *For each closed point $x \in X^{ss}$ that lies in a closed G-orbit, the isotropy representations of G_x on the $\mathcal{O}_X$-modules*
$$\mathcal{H}^{-j}\big(\mathbf{E} \otimes (\mathcal{O}_X/\mathfrak{M}_x)\big)$$
are trivial for all j.

In Section 4 we use this result to describe the derived category of $X/\!\!/G$ in terms of sheaves on X^{ss}. Our main result here offers a tool for making computations concerning $D(X/\!\!/G)$, which seems to be of significant interest in light of the increasing attention being paid of late to derived categories (see, for example, [Kon95]). Here we restrict ourselves to the case in which k is algebraically closed to simplify the statement.

Theorem 1.4. *Suppose X is a quasiprojective scheme over an algebraically closed field k of characteristic zero and that G is a reductive algebraic group over k that acts on X with a fixed choice of linearization. Then the bounded-above derived category $D^-(X/\!\!/G)$ of coherent sheaves on $X/\!\!/G$ is equivalent to the full triangulated subcategory of the bounded-above derived category of the category of G-equivariant coherent sheaves on X^{ss} that is generated by the complexes $\mathbf{E}$ satisfying the following condition:*

(†) *for each closed point $x \in X^{ss}$ that lies in a closed G-orbit of X^{ss}, the isotropy group G_x acts trivially on $\mathcal{H}^j\big(\mathbf{E} \overset{\mathbf{L}}{\otimes} (\mathcal{O}_{X^{ss}}/\mathfrak{M}_x)\big)$ for all $j \in \mathbf{Z}$.*

Note that a similar description holds for the bounded derived category if and only if $\big(\pi_*\mathcal{O}_{X^{ss}}\big)/\mathcal{O}_{X/\!\!/G}$ has finite Tor-dimension as an $\mathcal{O}_{X/\!\!/G}$-module.

Remark 1.5. It seems worth noting that Drezet–Narasimhan ([DN89]) state Theorem 1.1 for integral varieties X, although the proof works even when X is neither reduced nor irreducible; and indeed we are careful in the rest of the paper to ensure that all results are valid for arbitrary quasi-projective schemes. The reader who is familiar with the theory of linearizations of G-actions may wonder whether this is a useful generalization, since the most general theorem which allows one to deduce the existence of a G-linearization requires that the variety X be very well behaved (in particular, integral and even normal). However, there are many settings in which one has a *given* linearization of a reducible or nonreduced scheme—for example, Grothendieck's construction of Hilbert schemes and Quot schemes shows that they are equipped with ample line bundles which, one may easily check, can be made G-equivariant—and so in these cases the additional generality is essential.

We should at this point acknowledge some significant questions that are left open here. One of these is how to extend the results of this paper to varieties and groups over a field of finite characteristic. The author is hesitant even to guess what additional elements may be necessary to understand that setting, although undoubtedly the work of Haboush ([Hab75]) and of Nisnevič ([Nis77]) are relevant.

Another question concerns how the derived category of $X/\!\!/G$ relates to the G-equivariant derived category $D_G(X^{ss})$ of X^{ss} (which, one should note, need not be the derived category of *any* abelian category). Certainly one has a morphism $D(X/\!\!/G) \rightarrow D_G(X^{ss})$, but it would be very nice to have a description of the essential image of this morphism. In particular, such a description might make it possible to use the deformation theory of the stack-theoretic quotient $[X^{ss}/G]$ to study that of $X/\!\!/G$.

Section 5 is an appendix that contains many of the technical tools we need in the body of the paper; the technical facts are collected in that section both in order to streamline the proofs in Sections 2, 3, and 4 and to allow the curious reader to quickly see some of the elements of our analysis—for example, this would allow the reader interested in problems in characteristic p to get some sense of the extent to which our tools fail in that setting.

The author is grateful to Igor Dolgachev, discussions with whom were integral to this project, and who suggested a method for generalizing the results to non-algebraically closed fields; and to Mark Dickinson and Chris Skinner, who helped the author to understand semilinear group representations.

2. Proofs of Theorems 1.1 and 1.2

In this section we sketch, with a few changes, the proof of Theorem 1.1 that is given in [DN89], and then proceed in the second part to prove Theorem 1.2 by a simple extension of the argument used in Theorem 1.1.

2.1. **Proof of Theorem 1.1.** One direction is clear: if E descends to $X/\!\!/G$, then the isotropy representations of G_x on E_x at closed points $x \in X$ must be trivial. So, suppose the isotropy representations at closed points $x \in X$ that lie in closed orbits of G in X^{ss} are trivial; we want to show that E descends to the GIT quotient.

We may immediately reduce to the case in which $X = \operatorname{Spec} A$ is an affine scheme and $E = \widetilde{M}$ for some rank r projective A-module M. By Lemma 2.2 of [DN89], it is enough to show that $\operatorname{Spec} A$ is covered by G-invariant open sets over which E is G-equivariantly isomorphic to a free module of rank r.

Suppose $x \in X$ lies in a G-orbit that is not closed; then its closure contains a G-orbit of minimal dimension, which must therefore be closed (this is a standard fact for varieties, but the proof works equally well for nonintegral noetherian schemes). Suppose y is a point in this closed G-orbit. We will show that there is a G-invariant neighborhood of y in $\operatorname{Spec} A$ on which M is G-equivariantly isomorphic to a free module; because y lies in the closure of $G \cdot x$, this open set will contain some (hence every) point of $G \cdot x$ and will provide the necessary open set containing x.

Because G_y acts trivially on E_y, the G-equivariant vector bundle $E|_{(G \cdot y)_{\mathrm{red}}}$ on $(G \cdot y)_{\mathrm{red}} \cong G/G_y$ is trivial, and we may choose r G-invariant sections $\bar{s}_1, \ldots, \bar{s}_r$ trivializing $E|_{(G \cdot y)_{\mathrm{red}}}$. If $I \subset A$ is the ideal defining $(G \cdot y)_{\mathrm{red}} \subseteq X$, this means that $M \underset{A}{\otimes} A/I \cong \big(A/I\big)^r$ equivariantly via the sections $\bar{s}_1, \ldots, \bar{s}_r$. But now the A-module homomorphism

$$M \to M/IM$$

is surjective, and thus we may lift $\bar{s}_1, \ldots, \bar{s}_r$ to elements $s_1, \ldots, s_r$ of M.

Because G is reductive and acts rationally on A and M, there is a projection map $M \xrightarrow{p} M^G$, called the *Reynolds operator*, that commutes with the projection to M/IM: that is, the diagram

$$\begin{tikzcd}
M \arrow[r, "p"] \arrow[d] & M^G \arrow[d] \\
M/IM \arrow[r, "\bar{p}"] & \big(M/IM\big)^G
\end{tikzcd}$$

commutes. But now $\bar{p}(\bar{s}_i) = \bar{s}_i$ for each i, $1 \leq i \leq r$, and so the image of $p(s_i)$ in M/IM is $\bar{s}_i$ for every i.

Since $p(s_1), \ldots, p(s_r)$ generate M/IM, there is a G-invariant affine open subscheme of $\operatorname{Spec} A$ that contains $G \cdot y$ and over which $p(s_1), \ldots, p(s_r)$ span E as an $\mathcal{O}_X$-module. This gives the desired open set over which E is G-equivariantly isomorphic to a free rank r $\mathcal{O}_X$-module, completing the proof. □

2.2. **Proof of Theorem 1.2.** Our work will be simplified by the following.

Lemma 2.1. *Suppose $\overline{\mathcal{M}}$ is a coherent $\mathcal{O}_{X/\!\!/G}$-module for which $\pi^*\overline{\mathcal{M}}$ is G-equivariantly isomorphic to $\mathcal{M}$. Then $\overline{\mathcal{M}}$ is isomorphic to $(\pi_*\mathcal{M})^G$.*

Proof. The question is local, so we may reduce to the case $X = \operatorname{Spec} A$, G acting rationally on A, $\mathcal{M}$ a finitely presented A-module, $\overline{\mathcal{M}}$ a finitely presented A^G-module for which

$$\overline{\mathcal{M}} \underset{A^G}{\otimes} A \cong \mathcal{M}.$$

Claim 2.2.

$$\big(\overline{\mathcal{M}} \underset{A^G}{\otimes} A\big)^G \cong \overline{\mathcal{M}}.$$

Proof of Claim. We certainly have a map

$$\overline{\mathcal{M}} \to \big(\overline{\mathcal{M}} \underset{A^G}{\otimes} A\big)^G$$

of A^G-modules; it is enough to show that $\big(\bar{\mathcal{M}} \underset{A^G}{\otimes} A_G\big)^G = \{0\}$. Here A_G is the A^G-submodule of A of coinvariants: A splits both G-equivariantly and as an A^G-module as

$$A = A^G \oplus A_G$$

by Section 1.1 of [MFK94], so

$$\overline{\mathcal{M}} \underset{A^G}{\otimes} A = \big(\overline{\mathcal{M}} \underset{A^G}{\otimes} A^G\big) \oplus \big(\overline{\mathcal{M}} \underset{A^G}{\otimes} A_G\big) = \overline{\mathcal{M}} \oplus \big(\overline{\mathcal{M}} \underset{A^G}{\otimes} A_G\big).$$

But now $\overline{\mathcal{M}} \otimes A_G$ is, as a rational G-representation, a quotient of the rational G-representation $\overline{\mathcal{M}} \underset{k}{\otimes} A_G$. Because k is of characteristic zero and G is reductive, each splits (see Section 5.1) as a direct sum of irreducible representations, and since $\overline{\mathcal{M}}$ is a trivial G-representation, the isotypic components appearing in $\overline{\mathcal{M}} \underset{k}{\otimes} A_G$ are exactly those appearing in A_G. In particular, the trivial representation never appears in $\overline{\mathcal{M}} \underset{k}{\otimes} A_G$, hence it never appears in $\overline{\mathcal{M}} \underset{A^G}{\otimes} A_G$, proving the claim. $\square$

Now we obtain

$$\mathcal{M}^G \cong \big(\overline{\mathcal{M}} \underset{A^G}{\otimes} A\big)^G \cong \overline{\mathcal{M}},$$

proving the lemma. $\square$

By the lemma, then, condition (1) of Theorem 1.2 is equivalent to

$$\pi^*\big[(\pi_*\mathcal{M})^G\big] \cong \mathcal{M}. \tag{2.1}$$

Suppose first that $\mathcal{M}$ satisfies Equation (2.1); we will prove that condition (2) of Theorem 1.2 is satisfied. Again, we may work locally, and thus we assume that $X = \operatorname{Spec} A$ and that G acts rationally on A. Suppose that $x \in X$ is a point determined by a prime ideal $P \subset A$. Let G_x denote the isotropy subgroup of x; then G_x acts on A and fixes the ideal P.

By assumption, $\mathcal{M}$ is finitely presented over A, say with presentation

$$A^n \longrightarrow A^m \longrightarrow \mathcal{M} \longrightarrow 0,$$

and thus, since G is reductive, the induced sequence of G-invariants

$$\big(A^G\big)^n \longrightarrow \big(A^G\big)^m \longrightarrow \mathcal{M}^G \longrightarrow 0$$

gives a finite presentation of $\mathcal{M}^G$. Consequently, though, tensoring over A^G with A^{G_x} gives a finite presentation of $\mathcal{M}^{G_x} \overset{\text{def}}{=} \big(\mathcal{M}^G\big) \underset{A^G}{\otimes} A^{G_x}$. Moreover, localizing this sequence at $Q \overset{\text{def}}{=} P \cap A^{G_x}$ gives a finite presentation of $\big(\mathcal{M}^{G_x}\big)_Q$. Choose a *minimal* (see Section 5.2) free presentation

$$\big(A_Q^{G_x}\big)^{b_1} \xrightarrow{\psi} \big(A_Q^{G_x}\big)^{b_0} \to \big(\mathcal{M}^{G_x}\big)_Q \to 0 \tag{2.2}$$

of $\big(\mathcal{M}^{G_x}\big)_Q$ as an $A_Q^{G_x}$-module.

Let $\mathbf{F}$ denote the presentation given in Equation (2.2). Then $\mathbf{F} \underset{A_Q^{G_x}}{\otimes} A_P$ gives a free presentation of $\big(\mathcal{M}^{G_x}\big)_Q \underset{A_Q^{G_x}}{\otimes} A_P = \mathcal{M}_P$; moreover

$$\big(\mathbf{F} \underset{A_Q^{G_x}}{\otimes} A_P\big) \underset{A_P}{\otimes} \big(A_P/PA_P\big) = \mathbf{F} \underset{A_Q^{G_x}}{\otimes} \big(A_P/PA_P\big) \cong \Big(\mathbf{F} \underset{A_Q^{G_x}}{\otimes} \big(A_Q^{G_x}/Q\big)\Big) \underset{A_Q^{G_x}/Q}{\otimes} \big(A_P/PA_P\big).$$

Because $\mathbf{F}$ is minimal, we have that the map $\psi \underset{A_Q^{G_x}}{\otimes} \left(A_Q^{G_x}/Q\right)$ is zero, and thus that

$$\mathcal{M} \underset{A}{\otimes} \left(A_P/PA_P\right) \cong \left[\left(A_Q^{G_x}\right)^{b_0} \underset{A_Q^{G_x}}{\otimes} \left(A_Q^{G_x}/Q\right)\right] \underset{A_Q^{G_x}/Q}{\otimes} (A_P/PA_P)$$

G_x-equivariantly, while $\operatorname{Tor}_1^A\left(\mathcal{M}, A_P/PA_P\right)$ is a G_x-equivariant quotient of

$$\left[\left(A_Q^{G_x}\right)^{b_1} \underset{A_Q^{G_x}}{\otimes} \left(A_Q^{G_x}/Q\right)\right] \underset{A_Q^{G_x}/Q}{\otimes} (A_P/PA_P)\,.$$

Because these are generated over A_P by the images of the subspaces of G_x-invariant elements $\left(A_Q^{G_x}\right)^{b_0}$ and $\left(A_Q^{G_x}\right)^{b_1}$ respectively, condition (2) is satisfied.

Now, suppose that condition (2) holds. We will show that this implies condition (1).

Case 1. k algebraically closed. Certainly condition (2) implies condition (3) in this case, so we prove that condition (3) implies condition (1) provided that k is algebraically closed.

Choose a finite rank G-equivariant A-module F_0' equipped with a G-equivariant surjection $F_0' \to \mathcal{M}$. For example, one may split $\mathcal{M}$ into its G-isotypic components as a k-vector space representation, and choose elements $m_1, \ldots, m_r$ lying in G-isotypic components that generate $\mathcal{M}$. Then the isotypic components $V_1, \ldots, V_r$ containing $m_1, \ldots, m_r$ give one such A-module, namely

$$F_0' = \sum_{i=1}^{r} A \underset{k}{\otimes} V_i.$$

For each $x \in \operatorname{Spec} A$ lying in a closed G-orbit, one has a surjection

$$F_0' \otimes (A/\mathfrak{M}_x) \longrightarrow \mathcal{M} \otimes (A/\mathfrak{M}_x).$$

By assumption G_x acts trivially on $\mathcal{M} \otimes (A/\mathfrak{M}_x)$, and by Corollary 5.3 the isotropy group G_x is reductive by finite; hence by Corollary 5.7 there are G_x-invariant elements $s_1(x), \ldots, s_{r(x)}(x)$ of $F_0' \otimes (A/\mathfrak{M}_x)$ the images of which in $\mathcal{M} \otimes (A/\mathfrak{M}_x)$ span $\mathcal{M} \otimes (A/\mathfrak{M}_x)$. By Lemma 3.2, there are G-invariant lifts $\tilde{s}_1(x), \ldots, \tilde{s}_{r(x)}(x)$ in F_0' the images of which in $F_0' \otimes (A/\mathfrak{M}_x)$ are $s_1(x), \ldots, s_{r(x)}(x)$. The elements $\tilde{s}_1(x), \ldots, \tilde{s}_{r(x)}(x)$, moreover, generate $\mathcal{M} \otimes (A/\mathfrak{M}_y)$ for all y in an open set U_x of $\operatorname{Spec} A$ that contains x, and, because $\operatorname{Spec} A$ is quasicompact, we thereby obtain finitely many points $x_1, \ldots, x_p$ so that the set $\{\tilde{s}_u(x_v)\}$ of G-invariant elements of F_0' have images in $\mathcal{M}$ that generate $\mathcal{M}$. So we obtain a G-equivariant morphism

$$F_0 = \sum_{u,v} A \cdot \tilde{s}_u(x_v) \longrightarrow \mathcal{M}.$$

Let $\mathcal{K}$ denote its kernel.

For every $x \in \operatorname{Spec} A$ lying in a closed G-orbit, we have an exact sequence

$$\operatorname{Tor}_1^A\left(\mathcal{M}, (A/\mathfrak{M}_x)\right) \to \mathcal{K} \otimes (A/\mathfrak{M}_x) \to F_0 \otimes (A/\mathfrak{M}_x) \to \mathcal{M} \otimes (A/\mathfrak{M}_x) \to 0.$$

Because G_x acts trivially on $F_0 \otimes (A/\mathfrak{M}_x)$ by construction and on $\operatorname{Tor}_1^A\left(\mathcal{M}, (A/\mathfrak{M}_x)\right)$ by assumption, Corollary 5.7 implies that G_x acts trivially on $\mathcal{K} \otimes (A/\mathfrak{M}_x)$ for each such x. Repeating the above construction, this time for $\mathcal{K}$ instead of $\mathcal{M}$, we get a trivial G-representation V and a G-equivariant homomorphism

$$F_1 = A \underset{k}{\otimes} V \longrightarrow F_0$$

the image of which is $\mathcal{K}$. Thus, we have a G-equivariant presentation

$$F_1 \to F_0 \to \mathcal{M} \to 0$$

in which F_1 and F_0 descend to $\operatorname{Spec}(A^G)$. We obtain a commutative diagram

$$\begin{tikzcd}
(F_1^G) \underset{A^G}{\otimes} A \arrow[r] \arrow[d, "\wr"] & (F_0^G) \underset{A^G}{\otimes} A \arrow[r] \arrow[d, "\wr"] & (\mathcal{M}^G) \underset{A^G}{\otimes} A \arrow[r] & 0 \\
F_1 \arrow[r] & F_0 \arrow[r] & \mathcal{M} \arrow[r] & 0
\end{tikzcd}$$

with exact rows, implying that $\mathcal{M}^G \underset{A^G}{\otimes} A = \mathcal{M}$. This completes the proof when k is algebraically closed.

Case 2. ***k* arbitrary.** We now use the algebraically closed case to prove the result for arbitrary k. We will denote by $\bar{k}$ a fixed algebraic closure of k, and by $A_{\bar{k}}$, $\mathcal{M}_{\bar{k}}$ and so on the tensor products with $\bar{k}$. We will also let $\Gamma = \operatorname{Gal}(\bar{k}/k)$ denote the Galois group of $\bar{k}$ over k.

Let $\mathfrak{M}$ denote a maximal ideal of $A_{\bar{k}}$, and let $P = \mathfrak{M} \cap A$ denote its intersection with A. Let $x \in \operatorname{Spec} A_{\bar{k}}$ denote the closed point of $A_{\bar{k}}$ corresponding to $\mathfrak{M}$, and let $y \in \operatorname{Spec} A$ denote the point of $\operatorname{Spec} A$ corresponding to P.

Claim 2.3. One has an inclusion of isotropy groups

$$(G_{\bar{k}})_x \subseteq (G_y)_{\bar{k}}\,.$$

Proof of Claim. The $G_{\bar{k}}$-action on $\operatorname{Spec} A_{\bar{k}}$ covers the G-action on $\operatorname{Spec} A$, and so in particular the image of $(G_{\bar{k}})_x$ in G fixes y. ☐

We will show that, assuming condition (2) of Theorem 1.2, the isotropy condition (3) is satisfied for the $A_{\bar{k}}$-module $\mathcal{M}_{\bar{k}}$. Indeed, from Claim 2.3, it is enough to prove the following.

Claim 2.4. For any A-module N and any $i \geq 0$,

$$\operatorname{Tor}_i^A(N, A/P) \underset{A/P}{\otimes} \bar{k} = \operatorname{Tor}_i^{A_{\bar{k}}}(N_{\bar{k}}, A/\mathfrak{M})\,.$$

Proof of Claim. Let $R \longrightarrow N$ denote a free resolution of N. Then

$$\operatorname{Tor}_i^A(N, A/P) \underset{A/P}{\otimes} (A_P/PA_P) = \operatorname{Tor}_i^{A_P}(N_P, A_P/PA_P) = H_i\left(R \underset{A}{\otimes} A_P \underset{A_P}{\otimes} A_P/PA_P\right).$$

Now the functor of tensoring with $A_{\bar{k}}/\mathfrak{M}$ over A_P/PA_P is exact (since this is just a field extension) and so we get

$$\begin{aligned}
\operatorname{Tor}_i^A(N, A/P) \underset{A/P}{\otimes} \bar{k} &= \operatorname{Tor}_i^A(N, A/P) \underset{A/P}{\otimes} (A_P/PA_P) \underset{A_P/PA_P}{\otimes} A_{\bar{k}}/\mathfrak{M} = \\
H_i\left(R \underset{A}{\otimes} A_P \underset{A_P}{\otimes} (A_P/PA_P) \underset{A_P/PA_P}{\otimes} A_{\bar{k}}/\mathfrak{M}\right) &\cong H_i\left(R \underset{A}{\otimes} (A_{\bar{k}}/\mathfrak{M})\right) = \\
H_i\left(R_{\bar{k}} \underset{A_{\bar{k}}}{\otimes} (A_{\bar{k}}/\mathfrak{M})\right) &= \operatorname{Tor}_i^A(N, A/\mathfrak{M})
\end{aligned} \tag{2.3}$$

as desired. ☐

Combining the two claims, we find that $\mathcal{M}_{\bar{k}} \underset{A_{\bar{k}}}{\otimes} (A_{\bar{k}}/\mathfrak{M})$ and $\operatorname{Tor}_1^{A_{\bar{k}}}(\mathcal{M}_{\bar{k}}, A_{\bar{k}}/\mathfrak{M})$ are generated over $\bar{k}$ by elements invariant under $(G_{\bar{k}})_x$, and thus are in fact trivial representations of this isotropy group.

As a consequence, Case 1 implies that

$$(\mathcal{M}_{\bar{k}})^{G_{\bar{k}}} \underset{A_{\bar{k}}^{G_{\bar{k}}}}{\otimes} A_{\bar{k}} = \mathcal{M}_{\bar{k}}. \tag{2.4}$$

Recall that we want to show that, if $\mathcal{M}$ satisfies condition (2) of Theorem 1.2, then

$$\mathcal{M}^G \underset{A^G}{\otimes} A = \mathcal{M}.$$

Equation (2.4) implies that, to prove Case 2, it will be enough to show that

$$\left[(\mathcal{M}_{\bar{k}})^{G_{\bar{k}}} \underset{A_{\bar{k}}^{G_{\bar{k}}}}{\otimes} A_{\bar{k}}\right]^{\Gamma} = \mathcal{M}^G \underset{A^G}{\otimes} A.$$

Claim 2.5. If V is a rational G-representation defined over k, then

$$\left(V^G\right)_{\bar{k}} = (V_{\bar{k}})^{G_{\bar{k}}}.$$

Proof of Claim. Let S denote the coordinate ring of G (here we use that G is an affine algebraic group), and let

$$\sigma : V \longrightarrow S \underset{k}{\otimes} V$$

denote the dual action of G on V (see Section 1.1 of [MFK94]). Then the dual action of $G_{\bar{k}}$ on $V_{\bar{k}}$ is given by

$$\sigma \otimes \bar{k} : V_{\bar{k}} \longrightarrow S_{\bar{k}} \underset{\bar{k}}{\otimes} V_{\bar{k}},$$

and the space of invariants in $V_{\bar{k}}$, that is, the space of elements $\sum v_i \otimes \lambda_i$ in $V \otimes \bar{k}$ for which

$$\sum \sigma(v_i) \otimes \lambda_i = 1 \otimes \left(\sum v_i \otimes \lambda_i\right), \tag{2.5}$$

may be identified with the tensor product with $\bar{k}$ of the space of elements v of V for which $\sigma(v) = 1 \otimes v$, as desired: supposing, as one may, that $\lambda_1, \ldots, \lambda_k$ are linearly independent over k, this follows immediately from Equation 2.5. ☐

As a result, we have

$$\left[(\mathcal{M}_{\bar{k}})^{G_{\bar{k}}} \underset{A_{\bar{k}}^{G_{\bar{k}}}}{\otimes} A_{\bar{k}}\right]^{\Gamma} = \left[\left(\mathcal{M}^G\right)_{\bar{k}} \underset{(A^G)_{\bar{k}}}{\otimes} A_{\bar{k}}\right]^{\Gamma} = \left[\left(\mathcal{M}^G \underset{A^G}{\otimes} A\right)_{\bar{k}}\right]^{\Gamma} = \mathcal{M} \underset{A^G}{\otimes} A$$

as desired.

3. Proof of Theorem 1.3

We first mention that there is a simple technique for replacing G-equivariant coherent sheaves by G-equivariant vector bundles.

Lemma 3.1. *Suppose $\mathcal{M}$ is a G-equivariant coherent sheaf on a quasiprojective G-scheme Y equipped with a G-linearized polarization H. Then there is a finite rank G-equivariant vector bundle $\mathbf{V}$ and a surjective G-equivariant homomorphism $\mathbf{V} \to \mathcal{M}$.*

Proof. For each closed point $y \in Y$ one may choose elements $\overline{m}_1(y), \dots, \overline{m}_{n(y)}(y)$ of the fiber $\mathcal{M}_y$ that generate $\mathcal{M}_y$ over $\mathcal{O}_y$. These lift to local sections $m_1(y), \dots, m_{n(y)}(y)$ of $\mathcal{M}$ that generate $\mathcal{M}$ over some open set U_y of Y containing y. Because Y is quasiprojective it is quasicompact, and hence finitely many such open sets $U_{y_1}, \dots, U_{y_s}$ cover Y. Now choose an n sufficiently large that all the local sections in

$$\bigl\{m_i(y_j) \,\big|\, 1 \le j \le s, 1 \le i \le n(y_j)\bigr\}$$

extend to elements of $H^0\bigl(Y, \mathcal{M}(nH)\bigr)$. Because G acts rationally on $H^0\bigl(Y, \mathcal{M}(nH)\bigr)$, the elements of S generate a finite-dimensional G-invariant k-vector subspace V of $H^0\bigl(Y, \mathcal{M}(nH)\bigr)$. The natural homomorphism

$$\mathbf{V} = V \otimes \mathcal{O}(-nH) \longrightarrow \mathcal{M}$$

then has the desired property. □

Lemma 3.2. *Suppose* $\operatorname{Spec} A$ *is an affine G-scheme and $\mathcal{F}$ is a G-equivariant finitely generated projective A-module. Suppose $x \in$* $\operatorname{Spec} A$ *is a closed point, the associated ideal of which is $\mathfrak{M}_x$, the G-orbit through which is closed in* $\operatorname{Spec} A$ *and is defined by the ideal $I \subseteq A$. If $s \in \mathcal{F} \otimes A/\mathfrak{M}_x$ is G_x-invariant, then there is a G-invariant element $\tilde{s} \in \mathcal{F}$ such that the image of $\tilde{s}$ in $\mathcal{F} \otimes A/\mathfrak{M}_x$ is the element s.*

Proof. Pulling back the vector bundle associated to the projective module $\mathcal{F}$ to $\operatorname{Spec}(A/I)_{\mathrm{red}}$, we obtain a G-equivariant vector bundle on G/G_x, which must therefore be the vector bundle associated to some representation of G_x. Now s determines an element of this representation, and by hypothesis s lies in a trivial subrepresentation of G_x. The associated bundle of this subrepresentation, though, is a trivial bundle on G, and thus s extends to a G-invariant global section of the pullback of $\mathcal{F}$ to $G/G_x \cong \operatorname{Spec}(A/I)_{\mathrm{red}}$. But there is a G-equivariant surjection $\mathcal{F} \longrightarrow \mathcal{F} \otimes (A/I)_{\mathrm{red}}$, and because G is reductive, we may lift this element of $\mathcal{F} \otimes (A/I)_{\mathrm{red}}$ to a G-invariant element of $\mathcal{F}$. □

We will also need the following tools when k is not algebraically closed. As before, we let $\bar{k}$ denote an algebraic closure of k, we let $\Gamma = \mathrm{Gal}(\bar{k}/k)$, and we let $X_{\bar{k}}$, $\mathbf{E}_{\bar{k}}$, etc. denote the pullbacks of the relevant objects over $\operatorname{Spec} \bar{k}$.

Lemma 3.3. *Suppose $\mathbf{E}$ is a bounded above complex of G-equivariant vector bundles on X^{ss} that satisfies condition (2) of Theorem 1.3. Then the $G_{\bar{k}}$-equivariant complex $\mathbf{E}_{\bar{k}}$ on $X^{ss}_{\bar{k}}$ satisfies condition (3) of Theorem 1.3.*

Proof. The condition is local, so we may assume that $X^{ss} = \operatorname{Spec} A$. Let $\mathfrak{M} \subset A_{\bar{k}}$ denote a maximal ideal of $A_{\bar{k}}$, and let $P = \mathfrak{M} \cap A$. We have that

$$\mathcal{H}^{-i}\left(\mathbf{E}_{\bar{k}} \underset{A_{\bar{k}}}{\otimes} (A_{\bar{k}}/\mathfrak{M})\right) = \mathcal{H}^{-i}\left(\mathbf{E} \underset{A}{\otimes} (A_P/PA_P) \underset{A_P/PA_P}{\otimes} (A_{\bar{k}}/\mathfrak{M})\right) = \mathcal{H}^{-i}\left(\mathbf{E} \underset{A}{\otimes} (A_P/PA_P)\right) \underset{A_P/PA_P}{\otimes} (A_{\bar{k}}/\mathfrak{M})$$

because $A_P/PA_P \longrightarrow A_{\bar{k}}/\mathfrak{M}$ is a field extension. The last module is generated by $(G_y)_{\bar{k}}$-invariant elements by assumption, where y is the point of $\operatorname{Spec} A$ corresponding to P, and thus Claim 2.3 gives the desired conclusion for $(G_{\bar{k}})_x$ where x is the point in $\operatorname{Spec} A_{\bar{k}}$ corresponding to $\mathfrak{M}$. □

Lemma 3.4. *Let W denote a $\bar{k}$-vector subspace of a $\bar{k}$-vector space V with k-structure. Then W is defined over k if and only if W is invariant under the action of Γ on V.*

Proof. This is a special case of a proposition on page 30 of [Bor91]. □

3.1. Sufficiency of the Criterion.

Here we use the ideas of Section 2 to establish the sufficiency of the descent criterion of Theorem 1.3. Suppose

$$\mathbf{E}: \qquad \cdots \to E_n \xrightarrow{\phi_n} E_{n-1} \to \cdots \to E_2 \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0 \to 0$$

is a G-equivariant complex of vector bundles on X^{ss} as in the statement of Theorem 1.3. We construct, inductively in the homological degree of the complex $\mathbf{E}$, another G-equivariant complex $\mathbf{V}$ that is equivariantly quasi-isomorphic to $\mathbf{E}$ and that descends to $X/\!\!/G$.

The proof relies heavily on the following replacement technique.

Lemma 3.5. *Suppose*

$$\mathbf{E}: \qquad \cdots \to E_n \xrightarrow{\phi_n} E_{n-1} \to \cdots \to E_2 \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0 \to 0$$

is a G-equivariant complex of vector bundles on a quasiprojective scheme Y. Assume that, for some $n \in \mathbf{Z}$, one is given a G-equivariant vector bundle V on Y and a G-equivariant homomorphism $V \xrightarrow{f} E_n$ so that

1. $\operatorname{Im}(\phi \circ f) = \operatorname{Im}(\phi_n)$ *and*
2. *the induced homomorphism*

$$\operatorname{Ker}(\phi_n \circ f) \to \mathcal{H}^{-n}(\mathbf{E})$$

is surjective.

Then there is a G-equivariant complex $\mathbf{E}'$ of vector bundles and a G-equivariant quasi-isomorphism $\mathbf{E}' \xrightarrow{q} \mathbf{E}$ so that

1. *in degrees $j < n$ one has $E'_j = E_j$, and $q_j : E_j \to E_j$ is the identity map, and*
2. *in degree n the quasi-isomorphism q restricts to $V \xrightarrow{f} E_n$.*

Proof. We will work inductively: our starting data give a G-equivariant morphism of G-equivariant complexes of vector bundles

$$\begin{tikzcd}
 & V \arrow[r, "\phi_n\circ f"] \arrow[d, "f"] & E_{n-1} \arrow[r, "\phi_{n-1}"] \arrow[d] & E_{n-2} \arrow[r] \arrow[d] & \cdots \\
E_{n+1} \arrow[r, "\phi_{n+1}"] & E_n \arrow[r, "\phi_n"] & E_{n-1} \arrow[r, "\phi_{n-1}"] & E_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

which is a homology isomorphism in homological degrees less than or equal to $n-1$ and is surjective on homology in degree n. We will show that, given a diagram of complexes which induces a homology isomorphism in homological degrees less than or equal to $j-1$ and is surjective on homology in homological degree j, we may extend the top complex into homological degree $j+1$ so that the morphism so constructed is a homology isomorphism in homological degrees less than or equal to j and is surjective on homology in degree $j+1$; iterating this construction then builds the desired complex $\mathbf{E}'$ degree by degree.

So, suppose we are given a G-equivariant morphism

$$\begin{tikzcd}
 & E'_j \arrow[r, "\phi_n\circ f"] \arrow[d, "q_j"] & E'_{j-1} \arrow[r, "\phi'_{j-1}"] \arrow[d] & E_{j-2} \arrow[r] \arrow[d] & \cdots \\
E_{j+1} \arrow[r, "\phi_{j+1}"] & E_j \arrow[r, "\phi_j"] & E_{j-1} \arrow[r, "\phi_{j-1}"] & E_{j-2} \arrow[r] & \cdots
\end{tikzcd}$$

with the aforementioned properties. Let $\widetilde{E}'_{j+1}$ denote the fiber product of coherent sheaves

$$\widetilde{E}'_{j+1} = E_{j+1} \underset{E_j}{\times} E'_j;$$

this sheaf is the coherent subsheaf of $E_{j+1}\oplus E'_j$ consisting of pairs (e_1, e_2) of sections for which $\phi_{j+1}(e_1) = q_j(e_2)$. By construction $\widetilde{E}'_{j+1}$ is equipped with morphisms to E_{j+1} and E'_j that make the diagram

$$\begin{tikzcd}
\widetilde{E}'_{j+1} \arrow[r] \arrow[d] & E'_j \arrow[r, "\phi'_j"] \arrow[d, "q_j"] & E'_{j-1} \arrow[d, "q_{j-1}"] \\
E_{j+1} \arrow[r, "\phi_{j+1}"] & E_j \arrow[r, "\phi_j"] & E_{j-1}
\end{tikzcd} \tag{3.1}$$

G-equivariant and commutative. Furthermore, the induced morphism on homology in degree j is an isomorphism: by assumption the sheaf $\mathrm{Ker}(\phi'_j)$ surjects onto $\mathcal{H}^{-j}(\mathbf{E})$, and so it is enough to check that the image of $\widetilde{E}'_{j+1}$ in E'_j is the kernel of the map

$$\mathrm{Ker}(\phi'_j) \to \mathcal{H}^{-j}(\mathbf{E}).$$

A section e of $\mathrm{Ker}(\phi'_j)$ goes to zero in $\mathcal{H}^{-j}(\mathbf{E})$, however, exactly if there is some e' in E_{j+1} for which $\phi_{j+1}(e') = q_j(e)$, and then the section (e', e) lies in $\widetilde{E}'_{j+1}$ and maps to e.

In addition, it is easy to see that the kernel of the map

$$\widetilde{E}'_{j+1} \to E'_j \tag{3.2}$$

surjects onto $\operatorname{Ker}(\phi_{j+1})$ (if e is a section of the kernel, then $(e,0)$ lies in the kernel of (3.2)) and hence onto $\mathcal{H}^{-(j+1)}(\mathbf{E})$. Thus, to complete the proof it will be enough to produce a G-equivariant vector bundle E'_{j+1} and a G-equivariant surjective morphism of coherent sheaves $E'_{j+1} \to \widetilde{E}'_{j+1}$. But the existence of such a vector bundle and morphism is guaranteed by Lemma 3.1. This completes the proof of Lemma 3.5. □

We now proceed to prove the sufficiency of the criterion of Theorem 1.3, that is, that condition (2) implies condition (1); if k is algebraically closed, our construction for (1) will use only condition (3) rather than the full strength of condition (2). We replace $\mathbf{E}$ with a sequence of complexes $\mathbf{V}_j$ and quasi-isomorphisms

$$\cdots \to \mathbf{V}_2 \to \mathbf{V}_1 \to \mathbf{V}_0 \to \mathbf{E}$$

so that the complex $\mathbf{V}_j$ agrees with $\mathbf{V}_{j-1}$ in degrees less than j, and so that the bundles $(\mathbf{V}_j)_0$, $(\mathbf{V}_j)_1$, … , $(\mathbf{V}_j)_j$ all descend to $X/\!\!/G$; the inverse limit $\varprojlim \mathbf{V}_j$ is then our desired complex $\mathbf{V}$.

To construct $\mathbf{V}_0$, we proceed as follows. The surjection $(E_0)_{\bar{k}} \to \mathcal{H}^0(\mathbf{E}_{\bar{k}})$ yields a surjection $(E_0)_{\bar{k}} \otimes (\mathcal{O}_{X_{\bar{k}}}/\mathfrak{M}_x) \to \mathcal{H}^0(\mathbf{E}_{\bar{k}}) \otimes (\mathcal{O}_{X_{\bar{k}}}/\mathfrak{M}_x)$ for each closed point $x \in X^{ss}_{\bar{k}}$. By Lemma 3.3, the isotropy group $(G_{\bar{k}})_x$ acts trivially on

$$\mathcal{H}^0\big(\mathbf{E}_{\bar{k}} \otimes (\mathcal{O}_{X_{\bar{k}}}/\mathfrak{M}_x)\big) = \mathcal{H}^0(\mathbf{E}_{\bar{k}}) \otimes (\mathcal{O}_{X_{\bar{k}}}/\mathfrak{M}_x)$$

for each closed point $x \in X^{ss}_{\bar{k}}$ the $G_{\bar{k}}$-orbit through which is closed, and hence by Corollary 5.7 for each such $x \in X^{ss}_{\bar{k}}$ there are $(G_{\bar{k}})_x$-invariant elements $s_1(x), \ldots, s_{r(x)}(x)$ in $(E_0)_{\bar{k}} \otimes (\mathcal{O}_{X_{\bar{k}}}/\mathfrak{M}_x)$ the images of which in $\mathcal{H}^0(\mathbf{E}_{\bar{k}}) \otimes (\mathcal{O}_{X_{\bar{k}}}/\mathfrak{M}_x)$ generate $\mathcal{H}^0(\mathbf{E}_{\bar{k}}) \otimes (\mathcal{O}_{X_{\bar{k}}}/\mathfrak{M}_x)$. By Lemma 3.2 there is an open set $U_x \subseteq X^{ss}_{\bar{k}}$ containing x so that $s_1(x), \ldots, s_{r(x)}(x)$ may be lifted to $G_{\bar{k}}$-invariant sections of $(E_0)_{\bar{k}}$ over U_x, the images of which in $(E_0)_{\bar{k}} \otimes (\mathcal{O}_{X_{\bar{k}}}/\mathfrak{M}_x)$ are $s_1(x), \ldots, s_{r(x)}(x)$; by restricting U_x further if necessary we may assume that the images of these elements in $\mathcal{H}^0(\mathbf{E}_{\bar{k}})$ generate that coherent sheaf over U_x.

Because $X^{ss}_{\bar{k}}$ is quasicompact, there are finitely many points $x_1, \ldots x_p$ such that the open sets $U_{x_1}, \ldots, U_{x_p}$ cover $X^{ss}_{\bar{k}}$—the crucial point is that there is a closed $G_{\bar{k}}$-orbit in the closure of every $G_{\bar{k}}$-orbit, hence every point $y \in X^{ss}_{\bar{k}}$ is contained in some U_x. Choosing n sufficiently large, the sections $\tilde{s}_u(x_v)$ for $v = 1, 2, \ldots, p$ may be extended to sections of $(E_0)_{\bar{k}}(nH)$ over all of $X^{ss}_{\bar{k}}$, and, projecting them onto the space of $G_{\bar{k}}$-invariant sections of $(E_0)_{\bar{k}}(nH)$ if necessary (recall that the Reynolds operator will commute with the restriction map to each U_{x_v} and hence the $G_{\bar{k}}$-invariant projection of the extension of $\tilde{s}_u(x_v)$ will still restrict to the given section on U_{x_v}) we obtain $G_{\bar{k}}$-invariant sections, still denoted by $\tilde{s}_u(x_v)$, of $(E_0)_{\bar{k}}(nH)$ over $X^{ss}_{\bar{k}}$, the images of which in $\mathcal{H}^0(\mathbf{E}_{\bar{k}})$ generate $\mathcal{H}^0(\mathbf{E}_{\bar{k}})$ as an $\mathcal{O}_{X^{ss}_{\bar{k}}}$-module.

Now the Γ-orbits of these finitely many sections $\tilde{s}_u(x_v)$ of $(E_0)_{\bar{k}}(nH)$ are finite in $H^0\left(X^{ss}_{\bar{k}}, (E_0)_{\bar{k}}(nH)\right)$; hence there is a finite-dimensional Γ-invariant $\bar{k}$-vector subspace of $H^0\left(X^{ss}_{\bar{k}}, (E_0)_{\bar{k}}(nH)\right)^{G_{\bar{k}}}$ containing all the sections $\tilde{s}_u(x_v)$. By Lemma 3.4 this subspace is defined over k, and thus in particular it is generated by finitely many sections $\mathbf{v}_1, \ldots, \mathbf{v}_l$, that lie, by Claim 2.5, in $H^0\left(X^{ss}, E_0(nH)\right)^{G}$. These sections generate $\mathcal{H}^0\big(\mathbf{E}_{\bar{k}}(nH)\big)$ when pulled back to $X^{ss}_{\bar{k}}$, and consequently they must generate $\mathcal{H}^0\big(\mathbf{E}(nH)\big)$ over X^{ss} because $X^{ss}_{\bar{k}} \longrightarrow X^{ss}$ is faithfully flat.

Taking

$$(\mathbf{V}_0)_0 = \sum_u \mathcal{O}_{X^s s}(-nH)\cdot \mathbf{v}_u \longrightarrow E_0,$$

we obtain a G-equivariant vector bundle with a G-equivariant map to E_0 so that $(\mathbf{V}_0)_0$ surjects onto $\mathcal{H}^0(\mathbf{E})$ and $(\mathbf{V}_0)_0$ descends to $X/\!\!/G$; then, using Lemma 3.5, we extend $(\mathbf{V}_0)_0$ to a G-equivariant complex on X^{ss} equipped with the desired quasi-isomorphism to $\mathbf{E}$, thus giving the desired complex $\mathbf{V}_0$.

Next, suppose we have constructed $\mathbf{V}_j$. As above, let $x \in X^{ss}_{\bar{k}}$ denote a closed point the $G_{\bar{k}}$-orbit through which is closed, and choose a $G_{\bar{k}}$-invariant affine open neighborhood $\operatorname{Spec} A_{\bar{k}}$ of x. Localizing $A_{\bar{k}}$ at the idea $\mathfrak{M}_x$ corresponding to x (but continuing to denote the localized ring by $A_{\bar{k}}$), we have a local $\bar{k}$-algebra $A_{\bar{k}}$ on which $(G_{\bar{k}})_x$ acts rationally; moreover by Theorem 5.1, $(G_{\bar{k}})_x$ is reductive by finite, and hence by Corollary 5.10 the complex $(\mathbf{V}_j)_{\bar{k}} \otimes A_{\bar{k}}$ is $(G_{\bar{k}})_x$-equivariantly isomorphic to the direct sum of a minimal $(G_{\bar{k}})_x$-equivariant complex $\mathbf{M}$ and an acyclic $(G_{\bar{k}})_x$-equivariant complex $\mathbf{S}$. By assumption the bundles $(\mathbf{V}_j)_0, \dots , (\mathbf{V}_j)_j$ descend to $X/\!\!/G$, and hence the modules $\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_j$ and $\mathbf{M}_0$, $\mathbf{M}_1, \dots, \mathbf{M}_j$ are generated by $(G_{\bar{k}})_x$-invariant elements. Now, because $\mathbf{S}$ is acyclic and $\mathbf{M}$ is minimal, we have that

$$\mathcal{H}^{-(j+1)}\big(\mathbf{V}_{\bar{k}} \otimes (A_{\bar{k}}/\mathfrak{M}_x)\big) = \mathbf{M}_{j+1} \otimes (A_{\bar{k}}/\mathfrak{M}_x),$$

and as a result, by our assumption on the isotropy representations of $\mathbf{E}$ together with Lemma 3.3, the module $\mathbf{M}_{j+1} \otimes (A_{\bar{k}}/\mathfrak{M}_x)$ is generated by its $(G_{\bar{k}})_x$-invariant elements. Moreover, because $\mathbf{S}$ is split acyclic, we may $(G_{\bar{k}})_x$-equivariantly split the module $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1}$ as

$$\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1} = \mathbf{M}_{j+1} \oplus \operatorname{Im}(\mathbf{S}_{j+2}) \oplus \operatorname{Im}(\mathbf{S}_{j+1}).$$

This induces a $(G_{\bar{k}})_x$-equivariant splitting of $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1} \otimes (A_{\bar{k}}/\mathfrak{M}_x)$.

Choose $(G_{\bar{k}})_x$-invariant elements $s_1(x), \dots, s_r(x)$ of $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1} \otimes (A_{\bar{k}}/\mathfrak{M}_x)$ that form a $\bar{k}$-basis of the vector subspace $\big(\mathbf{M}_{j+1} \otimes (A_{\bar{k}}/\mathfrak{M}_x)\big) \oplus \big(\operatorname{Im}(\mathbf{S}_{j+1}) \otimes (A_{\bar{k}}/\mathfrak{M}_x)\big)$. By Lemma 3.2 these may be lifted to $G_{\bar{k}}$-invariant sections $\tilde{s}_1(x), \dots, \tilde{s}_r(x)$ of $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1}$ over $\operatorname{Spec} A_{\bar{k}}$, the images of which in $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1} \otimes (A_{\bar{k}}/\mathfrak{M}_x)$ are $s_1(x), \dots, s_r(x)$.

Consider the $A_{\bar{k}}$-submodule N of $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1}$ generated by $\tilde{s}_1(x), \dots, \tilde{s}_r(x)$. The free $A_{\bar{k}}$-module on generators $\tilde{s}_1(x), \dots, \tilde{s}_r(x)$ maps $G_{\bar{k}}$-equivariantly and surjectively onto N, and moreover it is of complementary rank to the free $A_{\bar{k}}$-submodule $\operatorname{Im}\big(\mathbf{S}_{j+2}\big)$ of $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1}$, and thus, since the sum of these two free $A_{\bar{k}}$-modules surjects onto $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1} \otimes (A_{\bar{k}}/\mathfrak{M}_x)$ by construction, the submodules N and $\operatorname{Im}\big(\mathbf{S}_{j+2}\big)$ form a direct sum decomposition of $\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1}$. But now this means both that $N \oplus \operatorname{Im}\big(\mathbf{S}_{j+2}\big)$ surjects onto $\operatorname{Im}\big(\left((\mathbf{V}_j)_{\bar{k}}\right)_{j+1}\big)$—and consequently that N must do so as well—and that the kernel of

$$N \oplus \operatorname{Im}\big(\mathbf{S}_{j+2}\big) \to \left((\mathbf{V}_j)_{\bar{k}}\right)_j,$$

which is just $\big(\ker(N \to \left((\mathbf{V}_j)_{\bar{k}}\right)_j)\big) \oplus \operatorname{Im}\big(\mathbf{S}_{j+2}\big)$, surjects onto $\mathcal{H}^{-(j+1)}\left((\mathbf{V}_j)_{\bar{k}}\right)$—and consequently that N must do so as well.

For each closed $x \in X^{ss}_{\bar{k}}$ the $G_{\bar{k}}$-orbit through which is closed, then, there are an open set U_x of $X^{ss}_{\bar{k}}$ containing x and sections $\tilde{s}_1(x), \ldots, \tilde{s}_{r(x)}(x)$ of $((\mathbf{V}_j)_{\bar{k}})_{j+1}$ over U_x such that the image of the composite

$$\sum_{i=1}^{r(x)} \mathcal{O}_{U_x} \cdot \tilde{s}_i(x) \to \big[\, ((\mathbf{V}_j)_{\bar{k}})_{j+1} \big]\big|_{U_x} \to \big[\, ((\mathbf{V}_j)_{\bar{k}})_j \big]\big|_{U_x} \tag{3.3}$$

equals $\operatorname{Im}\big(((\mathbf{V}_j)_{\bar{k}})_{j+1} \big|_{U_x} \to ((\mathbf{V}_j)_{\bar{k}})_j \big|_{U_x} \big)$ and the kernel of the composite (3.3) maps surjectively onto $\mathcal{H}^{-(j+1)}\left((\mathbf{V}_j)_{\bar{k}}\right)$. The open sets U_x form an open cover of $X^{ss}_{\bar{k}}$ since the closure of every $G_{\bar{k}}$-orbit contains a closed $G_{\bar{k}}$-orbit; hence we may choose finitely many points $x_1, \ldots, x_p$ so that $U_{x_1}, \ldots, U_{x_p}$ cover $X^{ss}_{\bar{k}}$.

Choose n sufficiently large that all the sections in $\big\{ \tilde{s}_u(x_v) \,\big|\, 1 \le v \le p \big\}$ extend to sections of $((\mathbf{V}_j)_{\bar{k}})_{j+1}(nH)$ over $X^{ss}_{\bar{k}}$, and choose $G_{\bar{k}}$-invariant extensions of these sections over all of $X^{ss}_{\bar{k}}$ (we may choose arbitrary extensions and project them onto $H^0\big(X^{ss}_{\bar{k}}, ((\mathbf{V}_j)_{\bar{k}})_{j+1}\big)^{G_{\bar{k}}}$ using the Reynolds operator, which commutes with restriction to each U_{x_v}); let $\tilde{s}_u(x_v)$ still denote the extensions as well. The Γ-orbits of these finitely many sections are finite; hence there is a finite-dimensional $\bar{k}$-vector subspace of $H^0\big(X^{ss}_{\bar{k}}, ((\mathbf{V}_j)_{\bar{k}})_{j+1}\big)^{G_{\bar{k}}}$ that is Γ-invariant and contains all the elements $\tilde{s}_u(x_v)$. By Lemma 3.4, then, this subspace is defined over k, and consequently by Claim 2.5 there are finitely many elements $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of $H^0\big(X^{ss}, (\mathbf{V}_j)_{j+1}\big)^{G}$, the images of which in $H^0\big(X^{ss}_{\bar{k}}, ((\mathbf{V}_j)_{\bar{k}})_{j+1}\big)^{G_{\bar{k}}}$ generate a $\bar{k}$-subspace containing all the original vectors $\tilde{s}_u(x_v)$.

Moreover, the natural map

$$\sum_u \mathcal{O}_{X^{ss}}(-nH) \cdot \mathbf{v}_u \longrightarrow (\mathbf{V}_j)_{j+1} \tag{3.4}$$

is G-equivariant and satisfies the conditions on the morphism f in Lemma 3.5. Indeed, by construction the pullback of this morphism to $X^{ss}_{\bar{k}}$ satisfies the given conditions over $X^{ss}_{\bar{k}}$ (note that we may have enlarged the relevant kernel, but the kernel of our new morphism (3.4) when pulled back to $X^{ss}_{\bar{k}}$ surjects onto the kernel of (3.3) and so the conditions are still satisfied). However the morphism $X^{ss}_{\bar{k}} \longrightarrow X^{ss}$ is faithfully flat, and so the conditions of 3.5 must also be satisfied on X^{ss}.

Consequently, there is a G-equivariant complex $\mathbf{V}_{j+1}$ of vector bundles and a G-equivariant quasi-isomorphism $\mathbf{V}_{j+1} \to \mathbf{V}_j$ so that

$$(\mathbf{V}_{j+1})_l = (\mathbf{V}_j)_l \qquad \text{for } l < j+1$$

and

$$(\mathbf{V}_{j+1})_{j+1} = \sum_u \mathcal{O}_{X^{ss}}(-nH) \cdot \mathbf{v}_u.$$

Because $(\mathbf{V}_{j+1})_{j+1}$ is a direct sum of copies of $\mathcal{O}_{X^{ss}}(-nH)$ and thus descends to X^{ss} by construction, this completes the inductive step.

3.2. **Necessity of the Criterion.**

Suppose

$$\mathbf{E}: \quad \cdots \to E_n \xrightarrow{\phi_n} E_{n-1} \to \cdots \to E_2 \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0 \to 0$$

is a G-equivariant bounded above complex of vector bundles on X^{ss} that descends to $X/\!\!/G$. Because the descent condition is local, we may assume that $X = \operatorname{Spec} A$ is affine and that $\mathbf{E}$ is a complex of projective A-modules.

Since $\mathbf{E}$ descends to $\operatorname{Spec} A$, we have that $\mathbf{E} = \mathbf{E}^G \underset{A^G}{\otimes} A$; hence for any prime ideal $P \subset A$ corresponding to a point $x \in \operatorname{Spec} A$ we have

$$\mathbf{E} \underset{A}{\otimes} (A_P/PA_P) = \mathbf{E}^G \underset{A^G}{\otimes} A \underset{A}{\otimes} (A_P/PA_P) = \mathbf{E}^G \underset{A^G}{\otimes} A_Q^{G_x} \underset{A_Q^{G_x}}{\otimes} \left(A_Q^{G_x}/Q\right) \underset{A_Q^{G_x}/Q}{\otimes} (A_P/PA_P), \tag{3.5}$$

where $Q = P \cap A^{G_x}$. Now $A_Q^{G_x}/Q \longrightarrow A_P/PA_P$ is a field extension, and so Equation (3.5) gives

$$\mathcal{H}^{-j}\big(\mathbf{E}\underset{A}{\otimes}(A_P/PA_P)\big) = \mathcal{H}^{-j}\left(\mathbf{E}^G \underset{A^G}{\otimes} A_Q^{G_x} \underset{A_Q^{G_x}}{\otimes} \left(A_Q^{G_x}/Q\right) \underset{A_Q^{G_x}/Q}{\otimes} (A_P/PA_P)\right) = \mathcal{H}^{-j}\left(\mathbf{E}^G \underset{A^G}{\otimes} A_Q^{G_x} \underset{A_Q^{G_x}}{\otimes} \left(A_Q^{G_x}/Q\right)\right) \underset{A_Q^{G_x}/Q}{\otimes} (A_P/PA_P). \tag{3.6}$$

The last term in Equation (3.6) is generated by the subspace

$$\mathcal{H}^{-j}\left(\mathbf{E}^G \underset{A^G}{\otimes} A_Q^{G_x} \underset{A_Q^{G_x}}{\otimes} \left(A_Q^{G_x}/Q\right)\right),$$

which consists of G_x-invariant elements. This completes the proof.

4. The Derived Category of $X/\!\!/G$

In this section, we assume that the field k is algebraically closed and of characteristic zero. We describe the derived category of $X/\!\!/G$ in terms of a derived category on X^{ss}. Definitions and machinery concerning the derived category may be found in [GM99] or [TT90].

Let

$$\pi : X^{ss} \longrightarrow X/\!\!/G$$

denote the projection morphism. Given a quasicoherent sheaf on X^{ss}, one may first apply π_* to it and then take the quasicoherent subsheaf on $X/\!\!/G$ consisting of G-invariant elements; we denote this composition of functors by π_*^G. Note that π_* is exact because π is an affine morphism, and the functor of G-invariants is exact because G is reductive; hence the composite π_*^G is exact as well.

Proposition 4.1. *The pair of functors*

$$\begin{tikzcd}\operatorname{Coh}(\mathcal{O}_{X/\!\!/G}) \arrow[r, shift left, "\pi^*"] & \operatorname{Coh}(G-\mathcal{O}_{X^{ss}}) \arrow[l, shift left, "\pi_*^G"]\end{tikzcd}$$

forms an adjoint pair between the categories of coherent sheaves on $X/\!\!/G$ and of G-equivariant coherent sheaves on X^{ss}.

Proof. The statement is local on the target $X/\!\!/G$, hence we reduce to the claim: if A is a k-algebra on which G acts rationally, then for any finite A-module M and finite A^G-module N one has

$$\mathrm{Hom}_{G,A}\big(N \underset{A^G}{\otimes} A, M\big) \cong \mathrm{Hom}_{A^G}\big(N, M^G\big).$$

This follows from the standard induction/restriction adjointness: the only difference here arises because, if $\psi : N \underset{A^G}{\otimes} A \to M$ is G-equivariant, then the image of N lies in M^G. □

Corollary 4.2. *One has an adjoint pair of functors* $\big(\mathbf{L}\pi^*, \pi_*^G\big)$ *between the associated derived categories* $D^-\big(\operatorname{Coh}(\mathcal{O}_{X/\!\!/G})\big)$ *and* $D^-\big(\operatorname{Coh}(G-\mathcal{O}_{X^{ss}})\big)$.

This is a purely formal consequence of Proposition 4.1: the usual adjoint pair $\big(\mathbf{L}\pi^*, \mathbf{R}\pi_*^G\big)$ may be replaced by $\big(\mathbf{L}\pi^*, \pi_*^G\big)$ because π_*^G is right exact.

Proposition 4.3. *The composition* $\pi_*^G \circ \mathbf{L}\pi^*$ *is an autoequivalence of the derived category of* $X/\!\!/G$. *The essential image of* $\mathbf{L}\pi^*$ *in* $D^-\big(\operatorname{Coh}(G-\mathcal{O}_{X^{ss}})\big)$ *is exactly that described in Theorem 1.4.*

Proof. If $\mathbf{M}$ is a bounded-above complex of coherent sheaves on $X/\!\!/G$, it may be replaced, up to quasi-isomorphism, by a bounded-above complex $\mathbf{F}$ of finite rank vector bundles on $X/\!\!/G$. Because the class of vector bundles on $X/\!\!/G$ is adapted to the functor π^* (see [GM99] for this terminology), we have that

$$\mathbf{L}\pi^*\mathbf{M} \cong \pi^*\mathbf{F},$$

and hence that $\pi_*^G\mathbf{L}\pi^*\mathbf{M} \cong \pi_*^G\pi^*\mathbf{F} = \mathbf{F}$, and similarly for morphisms. Hence $\pi_*^G \circ \mathbf{L}\pi^*$ is an autoequivalence of $D^-\big(\operatorname{Coh}(\mathcal{O}_{X/\!\!/G})\big)$.

It follows from Theorem 1.3 that a complex of vector bundles $\mathbf{E}$ in $D^-\big(\operatorname{Coh}(G-\mathcal{O}_{X^{ss}})\big)$ is quasi-isomorphic to a complex of vector bundles in the image of $\mathbf{L}\pi^*$ if and only if it satisfies condition (†) in Theorem 1.4, and thus that it lies in the essential image of $\mathbf{L}\pi^*$ if and only if it satisfies condition (†). This completes the proof. □

This proposition immediately yields Theorem 1.4.

Remark 4.4. As mentioned in the introduction, if $\big(\pi_*\mathcal{O}_{X^{ss}}\big)/\mathcal{O}_{X/\!\!/G}$ has finite Tor-dimension as an $\mathcal{O}_{X/\!\!/G}$-module, then one gets a similar isomorphism between the bounded derived category $D^b\big(\operatorname{Coh}(\mathcal{O}_{X/\!\!/G})\big)$ and the full triangulated subcategory of $D^b\big(\operatorname{Coh}(G-\mathcal{O}_{X^{ss}})\big)$ generated by those complexes satisfying condition (*) of Theorem 1.4. Indeed, any complex in $D^b\big(\operatorname{Coh}(\mathcal{O}_{X/\!\!/G})\big)$ is quasi-isomorphic to a complex $\mathbf{F}$ of vector bundles in the full triangulated subcategory of $D^-\big(\operatorname{Coh}(\mathcal{O}_{X/\!\!/G})\big)$ that consists of complexes with cohomologies in only finitely many degrees. Applying $\mathbf{L}\pi^*$ to this complex $\mathbf{F}$ of vector bundles, we obtain an object $\mathbf{L}\pi^*\mathbf{F}$ in $D^-\big(\operatorname{Coh}(G-\mathcal{O}_{X^{ss}})\big)$ which, by assumption, again has nontrivial cohomologies in only finitely many degrees. Hence the complex $\mathbf{L}\pi^*\mathbf{F}$ lies in a subcategory of $D^-\big(\operatorname{Coh}(G-\mathcal{O}_{X^{ss}})\big)$ that is equivalent to $D^b\big(\operatorname{Coh}(G-\mathcal{O}_{X^{ss}})\big)$, and we obtain a functor $\mathbf{L}\pi^*$ from $D^b\big(\operatorname{Coh}(\mathcal{O}_{X/\!\!/G})\big)$ to $D^b\big(\operatorname{Coh}(G-\mathcal{O}_{X^{ss}})\big)$.

5. Appendix: Facts about Group Actions

In what follows, we treat as standard the following facts about reductive groups and their representations in characteristic zero. We call an algebraic group *reductive by finite* if it is an extension of a finite group by a reductive group, that is, its identity component is reductive and the component group is finite.

Theorem 5.1. (see [Mat60] and [Mat61] or [Nis77]) *Suppose G is a reductive algebraic group over an algebraically closed field k of characteristic $p \geq 0$, and H is a closed k-subgroup of G. If the quotient scheme G/H is affine, then the identity component of H is reductive.*

One may combine the theorem with the following useful fact.

Proposition 5.2. (see page 53 of [Hum75]) *If H is an algebraic group, the identity component of H is a normal subgroup of finite index in H.*

Corollary 5.3. *Suppose a reductive group G over k acts on an affine variety X over k, and the closed point $x \in X$ has closed G-orbit in X. Then the isotropy group G_x is reductive by finite.*

We also need the following standard definition of a quotient arising in geometric invariant theory.

Definition 5.4. Let G be an affine algebraic group over k acting on a k-scheme X. A morphism $\phi : X \to Y$ is called a *good quotient* (or, in this paper, a GIT quotient) if

- ϕ is affine and G-invariant,
- ϕ is surjective, and $U \subset Y$ is open if and only if $\phi^{-1}(U) \subset X$ is open,
- the natural homomorphism $\mathcal{O}_Y \to \left(\phi_* \mathcal{O}_X\right)^G$ is an isomorphism,
- if W is an invariant closed subset of X, then $\phi(W)$ is a closed subset of Y; if W_1 and W_2 are disjoint invariant closed subsets of X, then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

5.1. Complete Reducibility of Representations.

Definition 5.5. Let $k \subseteq K$ be an algebraic extension of fields of characteristic zero. Let H denote a group equipped with a representation

$$H \xrightarrow{\sigma} \mathrm{Gal}(K/k).$$

An action of H on a K-vector space W is called *K-semilinear* if, for every $h \in H$, every $c \in K$, and every $w_1, w_2 \in W$, one has

$$h \cdot (cw_1 + w_2) = \sigma(h)(c)(h \cdot w_1) + (h \cdot w_2).$$

A K-semilinear representation W is *trivial* if $\operatorname{Ker} \sigma \subset H$ acts trivially on W; otherwise W is a *nontrivial* K-semilinear representation.

Notice that H need not act trivially on a trivial K-semilinear representation according to our definition—indeed, if the representation σ in the Galois group is nontrivial, H cannot.

As in other sections of the paper, we will refer to a group that is an extension of a finite group by a reductive algebraic group over k as a *reductive by finite k-group.*

Proposition 5.6. *Let k denote a field of characteristic zero. Suppose H is a reductive by finite k-group equipped with a rational action on the finite algebraic* *extension K over k. Suppose H acts K-semilinearly and k-rationally on the finite-dimensional K-vector space W. Then W splits uniquely as a direct sum of the K-vector subspace generated by W^H and the nontrivial irreducible K-semilinear subrepresentations of W.*

Proof. By assumption, H is equipped with a rational representation

$$H \xrightarrow{\sigma} \mathrm{Gal}(K/k)$$

in the finite group $\mathrm{Gal}(K/k)$; hence the kernel of σ contains the identity component of H, and in particular the kernel

$$H^\circ = \mathrm{Ker}\left(H \xrightarrow{\sigma} \mathrm{Gal}(K/k)\right)$$

is again a reductive by finite k-group.

Step 1. *The Nontrivial Part of the H° Action.* If H° acts trivially on W, then $W^{H^\circ} = W$ and one proceeds immediately to Step 2. Otherwise, suppose the kernel H° acts nontrivially on W. Because H° is reductive by finite and acts K-linearly on W, the representation W splits as a direct sum of the H°-fixed K-vector subspace W^{H° and the irreducible K-linear subrepresentations of W, say

$$W = W^{H^\circ} \bigoplus \big(\bigoplus_\mu W_\mu\big). \tag{5.1}$$

In addition, for each $h \in H$ and nontrivial K-semilinear representation W_μ, the k-vector subspace $h \cdot W_\mu$ of W is in fact an irreducible H° representation in a K-vector space: first, $h \cdot W_\mu$ is a K-vector space, since, for every $h \cdot w_1$ and $h \cdot w_2$ in $h \cdot W_\mu$ and every $c \in K$, one has

$$chw_1 + hw_2 = h\big(h^{-1}(c)w_1\big) + hw_2 = h\big(h^{-1}(c)w_1 + w_2\big).$$

Now W_μ is a K-vector subspace of W, so $h^{-1}(c)w_1 + w_2$ lies in W_μ, and thus $chw_1 + hw_2$ lies in $h \cdot W_\mu$, as desired.

Furthermore, if $h' \in H^\circ$, then

$$h'h \cdot W_\mu = h(h^{-1}h'h) \cdot W_\mu = h \cdot W_\mu$$

because, H° being normal in H, the element $h^{-1}h'h$ lies in H° and so takes W_μ to itself. Thus H° acts on $h \cdot W_\mu$. But, since $h \cdot W_\mu$ will be irreducible if and only if W_μ is, we find that $h \cdot W_\mu$ is also one of the nontrivial representations in the decomposition (5.1). Consequently for fixed μ the sum $\sum\limits_{h\in H} h \cdot W_\mu$ is a finite sum of irreducible K-vector space representations of H°, and is, by construction, a nontrivial K-semilinear irreducible representation of H as well. The sums of H-orbits of K-linear representations of H° thus provide a decomposition of $\bigoplus\limits_\mu W_\mu$ into nontrivial K-semilinear irreducible subrepresentations of W.

Step 2. *Structure of W^{H°.* The kernel H° acts trivially on W^{H°; hence the action of H on W^{H° factors through the quotient group H/H°, which itself injects into $\mathrm{Gal}(K/k)$. The discussion of Borel (see pages 30–31 of [Bor91]) then guarantees that the space of invariants $W^H = (W^{H^\circ})^H$ generates W^{H° as a K-vector space, completing the desired decomposition. □

Corollary 5.7. *Suppose H is a reductive by finite k-group equipped with a rational action on the finite algebraic extension K over k. Suppose H acts K-semilinearly and k-rationally on the finite-dimensional K-vector spaces W_1, W_2, and W_3. Suppose*

$$0 \to W_1 \to W_2 \to W_3 \to 0 \tag{5.2}$$

is a short exact sequence of K-vector spaces that is H-equivariant. Then the short exact sequence (5.2) *admits a splitting that is simultaneously K-linear and H-equivariant.*

Proof. By Proposition 5.6, all of W_1, W_2, and W_3 split uniquely as in the proposition. Furthermore, the inclusion $W_1 \subseteq W_2$ being H-equivariant, as a subspace of W_2 the representation W_1 must be a direct sum of some of the nontrivial K-semilinear irreducible representations of H in W_2 and a subspace of $W_2^{H^\circ}$. The short exact sequence 5.2 must therefore decompose H-equivariantly and K-linearly as a direct sum of two subsequences:

1. a short exact sequence of nontrivial K-semilinear irreducible representations of H, and
2. a short exact sequence of K-semilinear representations of H that are generated by H-invariant elements.

The first sequence splits canonically since its kernel consists of the direct sum of some subset of the direct summands in the canonical decomposition $\bigoplus\limits_\mu (W_2)_\mu$ of the nontrivial part of the H° decomposition of W_2. Moreover, the second sequence splits K-linearly and H-equivariantly as well: the sequence

$$0 \to W_1^H \to W_2^H \to W_3^H \to 0$$

is an exact sequence of K^H-vector spaces because H is reductive, and by Proposition 5.6 and Proposition 14.2 of [Bor91], we have also that $W_i^H \underset{K^H}{\otimes} K = W_i^{H^\circ}$ for each i; but any splitting of the sequence (5.1) will yield an H-equivariant and K-linear splitting of the sequence

$$0 \to W_1^{H^\circ} \to W_2^{H^\circ} \to W_3^{H^\circ} \to 0,$$

as desired. □

5.2. **Minimal Complexes.** The reader may wish to consult [Eis95] for material on acyclic complexes that is closely related to the presentation here.

Definition 5.8. A complex

$$\mathbf{F} : \qquad \cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to \cdots$$

of (not necessarily free) modules over a local ring $(A, \mathfrak{M})$ is *minimal* if the maps in the complex $\mathbf{F} \otimes (A/\mathfrak{M})$ are all zero; that is, for each i, the image of

$$\phi_i : F_i \to F_{i-1}$$

lies in $\mathfrak{M}F_{i-1}$.

Proposition 5.9. *Let k denote a field of characteristic zero. Suppose $(A, \mathfrak{M})$ is a local k-algebra with rational action by an algebraic group H over k that is reductive by finite; suppose further that the field $K := A/\mathfrak{M}$ is finite over k. Assume that, for every finitely generated free A-module F equipped with a rational H-action and* *every H-invariant direct summand $V \subseteq F \otimes K$ of $F \otimes K$, there is an H-invariant direct summand $F' \subseteq F$ of F for which $F' \otimes K = V$. Suppose that*

$$\mathbf{F}: \qquad \cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to 0$$

is a bounded above H-equivariant complex of H-equivariant finitely generated free A-modules. Then $\mathbf{F}$ is H-equivariantly isomorphic to the direct sum of a minimal H-equivariant complex $\mathbf{M}$ and a split acyclic H-equivariant complex $\mathbf{S}$.

Proof. We will construct the minimal H-equivariant subcomplex $\mathbf{M}$ and the acyclic subcomplex $\mathbf{S}$. First, to construct $\mathbf{S}$, consider the complex

$$\mathbf{F} \otimes K: \qquad \cdots \to F_2 \otimes K \xrightarrow{\phi_2 \otimes K} F_1 \otimes K \xrightarrow{\phi_1 \otimes K} F_0 \otimes K \to 0.$$

We have H-invariant K-vector subspaces $I_j = \operatorname{Im}(\phi_{j+1} \otimes K)$ of $F_j \otimes K$ for all j, and by Corollary 5.7 we may choose for each j an H-invariant K-vector subspace $\tilde{I}_j$ of $F_{j+1} \otimes K$ for which the restriction

$$(\phi_{j+1} \otimes K)\big|_{\tilde{I}_j} : \tilde{I}_j \to F_j \otimes K$$

is an isomorphism onto I_j. By assumption, we may choose for each j an H-invariant direct summand $\tilde{S}_j$ of F_{j+1} for which

$$\tilde{S}_j \otimes K = \tilde{I}_j, \tag{5.3}$$

and we set

$$S_j = \big(\phi_{j+1}(\tilde{S}_j)\big) \oplus \big(\tilde{S}_{j-1}\big).$$

These modules will give the factors of the subcomplex $\mathbf{S}$ of $\mathbf{F}$. We will also choose complementary H-invariant A-submodules M_j of F_j (that is, submodules M_j so that $F_j = M_j \oplus S_j$) so that $\phi_j(M_j) \subseteq M_{j-1}$ for all j; then the subcomplex $\mathbf{M}$ so defined will automatically be minimal and we will have $\mathbf{F} = \mathbf{M} \oplus \mathbf{S}$ exactly as desired. Some care is required in our choices of the modules M_j, however, and we will proceed to explain how these choices can be made.

We proceed by induction on j: in particular, we start by choosing M_0. By Corollary 5.7 there is an H-invariant complement $I_0^\perp$, and, choosing one such, we may by assumption lift it to an H-invariant direct summand M_0 of F_0 for which $M_0 \otimes K = I_0^\perp$. Now $F_0 = M_0 \oplus S_0$.

Given choices of $M_0, M_1, \dots M_j$ which form the terms of a subcomplex $\mathbf{M}$ of $\mathbf{F}$ up through homological degree j, in such a way that the truncated complex $\sigma_{\geq j}\mathbf{F}$ is isomorphic to the direct sum of the truncation $\sigma_{\geq j}\mathbf{S}$ and the complex determined by the modules M_i through degree j, we form the composite map

$$F_{j+1} \xrightarrow{\phi_{j+1}} F_j \xrightarrow{\operatorname{proj}_{S_j}} S_j,$$

the kernel of which we denote by K_{j+1}; here the second map is the projection onto S_j determined by the complementary A-module M_j. Since M_j and S_j are H-invariant the kernel K_{j+1} is an H-invariant A-submodule of F_{j+1}, and since $\tilde{S}_j$ maps isomorphically onto $\phi_{j+1}(\tilde{S}_j)$ under the morphism, we have that $F_{j+1} = K_{j+1} \oplus \tilde{S}_j$. Moreover, $\phi_{j+2}(\tilde{S}_{j+1}) \subseteq K_{j+1}$ simply because $\mathbf{F}$ is a complex.

In fact, $\phi_{j+2}(\tilde{S}_{j+1})$ is a direct summand of K_{j+1}: since K_{j+1} is a direct summand of F_{j+1} it is, in particular, a free A-module that satisfies $K_{j+1} \otimes K = \operatorname{Ker}(\phi_{j+1} \otimes K)$. By Corollary 5.7 we may choose an H-invariant complement $I_{j+1}^\perp$ to I_{j+1}

in $K_{j+1} \otimes K$, and then any lift of $I_{j+1}^{\perp}$ to an H-invariant direct summand M_{j+1} of K_{j+1} gives a complement to $\phi_{j+2}(\tilde{S}_{j+1})$ in K_{j+1} which, moreover, maps into M_j under ϕ_{j+1} (because K_{j+1} does by assumption). This completes the inductive step. □

Corollary 5.10. *Suppose k is an algebraically closed field of characteristic zero. Suppose $(A, \mathfrak{M})$ is the localization of a finitely generated k-algebra at a maximal ideal $\mathfrak{M}$, which is equipped with a rational action by an algebraic group H over k that is reductive by finite. Suppose that*

$$\mathbf{F}: \qquad \cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to 0$$

is a bounded above H-equivariant complex of H-equivariant finitely generated free A-modules. Then $\mathbf{F}$ is H-equivariantly isomorphic to the direct sum of a minimal H-equivariant complex $\mathbf{M}$ and a split acyclic H-equivariant complex $\mathbf{S}$.

Proof. This follows immediately from Proposition 5.9 because, when $A/fM = k$, for any finitely generated free A-module F one has

$$F \cong (F \otimes k) \underset{k}{\otimes} A$$

as H-equivariant A-modules. □

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