

REPRESENTABILITY FOR SOME MODULI STACKS OF FRAMED SHEAVES

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Introduction. Moduli problems for various kinds of framed sheaves have been studied and used in many settings (see, for example, [Tha94], [Bra91], [Nak94]), and there is a good general theory of moduli for semistable framed sheaves, thanks to the work of Huybrechts and Lehn ([HL95a], [HL95b]). By contrast, there seem to be only a few examples in which the *full* moduli functor for framed sheaves (without conditions of semistability) is known to be represented by a scheme. In this paper, we prove a representability theorem for the full moduli functors of framed torsion-free sheaves on projective surfaces under certain conditions.

Let S denote a smooth, connected complex projective surface, and let $D \subset S$ denote a smooth connected complete curve in S . Fix a vector bundle E on D . An *E -framed torsion-free sheaf on S* is a pair (\mathcal{E}, ϕ) consisting of a torsion-free sheaf \mathcal{E} on S and an isomorphism $\phi : \mathcal{E}|_D \rightarrow E$; the isomorphism ϕ is called an *E -framing* of \mathcal{E} . An *isomorphism* of E -framed torsion-free sheaves on S is an isomorphism of the underlying torsion-free sheaves on S that is compatible with the framings. Let $\mathrm{TF}_S(E)$ denote the moduli functor for isomorphism classes of E -framed torsion-free sheaves on S . The reader should note that in the work of Huybrechts–Lehn the framing ϕ need *not* be an isomorphism; as a consequence of our more restrictive definition, the moduli functors that we study have no hope of being proper.

Suppose the vector bundle E satisfies

$$(1) \quad H^0\left(D, \mathrm{End} E \otimes N_{D/S}^{-k}\right) = 0$$

for all $k \geq 1$; here $N_{D/S}$ is the normal bundle of D in S . If $D \subset S$ is an arbitrary curve, there may be very few such bundles. However, if D is smooth and has positive self-intersection in S , then $N_{D/S}^{-1}$ is a negative line bundle on D , and consequently this condition on E is an open condition which is satisfied by all semistable vector bundles on D .

Theorem 1. *Suppose that D has positive self-intersection in S and that the vector bundle E satisfies Condition (1) for all $k \geq 1$. Then the functor $\mathrm{TF}_S(E)$ is represented by a scheme.*

In the proof of Theorem 1 we work in the slightly more general setting of a family of vector bundles on D , parametrized by a scheme U , that satisfies Condition (1) for all $k \geq 1$ at every point of U . Note also that the reader who is familiar with the language of stacks may restate Theorem 1 in the following form: over the open substack of its target that parametrizes vector bundles on D that satisfy Condition (1) for all $k \geq 1$, the fibers of the restriction morphism from the moduli stack of

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torsion-free sheaves on S that are locally free along D to the moduli stack of vector bundles on D are schemes.

Functors of the type we study here arose naturally (in some special cases) in the representation-theoretic constructions of Nakajima; Theorem 1 demonstrates that the existence of the fine moduli schemes used by Nakajima is a much more general phenomenon, one which we hope can be exploited more widely in the study of sheaves on noncompact surfaces. The new ingredient in our proof of Theorem 1 is the use of formal geometry along the curve D ; in particular, the techniques used here are completely different from those of [HL95a], [HL95b], and make no use of geometric invariant theory (GIT). Although Lehn ([Leh93]) has, under some conditions on the curve D and the bundle E along the curve, proven that the full moduli functors for vector bundles on S with framing along D by E are represented by *algebraic spaces*, from the point of view of the usual GIT techniques it is perhaps surprising that there is a fine moduli *scheme* (a much stronger fact) for all framed sheaves: indeed, there can be framed sheaves that are not semistable for *any* polarization.

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Affine bundles over $\text{Bun}^\circ(D)$. In this section we construct the fundamental affine bundles \mathbf{A}_n (for n in the range $1 \leq n < \infty$) over U that we will use to embed the functor $\text{TF}_S(E)$ in a scheme. The construction of these bundles and the description of the universal properties they possess must be well known (cf. [Gri66], in which the relevant cohomology groups are discussed), but the author does not know a suitable reference.

Fix a surface S , a curve D in S , a scheme U , and a vector bundle E on $D \times U$ as in Theorem 1. Let $D^{(n)}$ (that is, D with structure sheaf \mathcal{O}_S/I_D^{n+1} , $0 \leq n < \infty$) denote the n th order neighborhood of D in S .

Definition 2. Let \mathcal{A}_n denote the moduli functor over U of isomorphism classes of triples $(\mathcal{E}, V \xrightarrow{f} U, \phi)$ consisting of

1. a vector bundle \mathcal{E} on $D^{(n)} \times V$,
2. a morphism $f : V \rightarrow U$, and
3. an isomorphism $\phi : \mathcal{E}|_{D \times V} \rightarrow (1_D \times f)^*E$.

Suppose that \mathcal{E} is a vector bundle over $D^{(n)}$; then \mathcal{E} has a canonical (decreasing) filtration as an $\mathcal{O}_{D^{(n)}}$ -module with filtered pieces $F_j\mathcal{E} = I_D^j\mathcal{E}$, where I_D is the ideal of $D \subset D^{(n)}$. By its construction, this filtration is preserved by any endomorphism of the vector bundle \mathcal{E} , and moreover $F_j\mathcal{E}/F_{j+1}\mathcal{E} \cong N_{D/S}^{-j} \otimes (F_0\mathcal{E}/F_1\mathcal{E})$ provided $0 \leq j \leq n$. Using these facts together with the exact sequence

$$0 \rightarrow \text{Hom}(E, E \otimes N_{D/S}^{-n}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{E}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{E}|_{D^{(n-1)}}) \rightarrow 0$$

and condition (1), one may prove by induction on n that $\text{End}(\mathcal{E}) \subseteq \text{End}(\mathcal{E}|_D)$ and consequently that E -framed bundles on $D^{(n)}$ are rigid.

Evidently $\mathcal{A}_0 \cong U$; moreover, there are maps $\pi_{n+1} : \mathcal{A}_{n+1} \rightarrow \mathcal{A}_n$ for all $n \geq 0$.

Proposition 3. *Each \mathcal{A}_n ($n \geq 1$) is represented by a scheme \mathbf{A}_n that is an affine bundle over \mathbf{A}_{n-1} .*

Proof. Working inductively, it will suffice to construct an \mathbf{A}_{n-1} -scheme \mathbf{A}_n that represents \mathcal{A}_n and is an affine bundle over \mathbf{A}_{n-1} . Fix a universal bundle $E^{(n-1)}$ on $D^{(n-1)} \times \mathbf{A}_{n-1}$. For any scheme T , an element of $\mathcal{A}_n(T)$ determines a map $f : T \rightarrow \mathbf{A}_{n-1}$, and, if (\mathcal{E}, ϕ) is the given element of $\mathcal{A}_n(T)$, there is an isomorphism of $\mathcal{E}|_{D^{(n-1)} \times T}$ with $(1 \times f)^* E^{(n-1)}$ compatibly with the framings by E . But then, because E -framed bundles on $D^{(n-1)}$ are rigid, we find that \mathcal{A}_n as a functor over \mathbf{A}_{n-1} is isomorphic to the functor taking $f : T \rightarrow \mathbf{A}_{n-1}$ to the set of isomorphism classes of pairs (\mathcal{E}, ϕ) consisting of a bundle \mathcal{E} on $D^{(n)} \times T$ together with an isomorphism ϕ of $\mathcal{E}|_{D^{(n-1)} \times T}$ with $(1 \times f)^* E^{(n-1)}$. We will refer to such a pair as an $E^{(n-1)}$ -framed bundle.

Because the statement of the proposition is local on \mathbf{A}_{n-1} , we may assume that \mathbf{A}_{n-1} is an affine scheme that is the spectrum of a local ring R . Since

$$H^2 \left(D \times \mathbf{A}_{n-1}, \text{End } E \otimes N_{D/S}^{-n} \right) = 0,$$

there is no obstruction to extending $E^{(n-1)}$ to a bundle on $D^{(n)} \times \mathbf{A}_{n-1}$ (see [Gri66]); for convenience, fix such an extension V together with an isomorphism of the restriction $V|_{D^{(n-1)} \times \mathbf{A}_{n-1}}$ with $E^{(n-1)}$.

Choose a cover $\{W_\alpha\}$ of $D^{(n)}$ by affine open sets. Fix an affine \mathbf{A}_{n-1} -scheme $f : T \rightarrow \mathbf{A}_{n-1}$. Note that the given framing of V by $E^{(n-1)}$, and hence by E , induces a canonical isomorphism of $F_n V$ with $E \otimes N_{D/S}^{-n}$. Given a 1-cocycle $(e_{\beta\alpha})$ in $Z^1(\{W_\alpha \times T\}, \text{Hom}(V, E \otimes N_{D/S}^{-n}))$, one may define a vector bundle on $D^{(n)} \times T$ with $E^{(n-1)}$ -framing by using as gluing data the $\underline{\text{Aut}}(V)$ -valued cocycle $(1 + e_{\beta\alpha})$ (which may be taken to be $\underline{\text{Aut}}(V)$ -valued because $\text{Hom}(V, E \otimes N_{D/S}^{-n})$ is canonically isomorphic to $\text{Hom}(V, F_n V)$); note that, because this cocycle reduces to the identity on $D^{(n-1)}$, the bundle it defines comes equipped with an $E^{(n-1)}$ -framing, as desired. This gives a method of associating to each element of $Z^1(\{W_\alpha \times T\}, \text{Hom}(V, E \otimes N_{D/S}^{-n}))$ an $E^{(n-1)}$ -framed vector bundle on $D^{(n)} \times T$. It is then straightforward to check the following by hand.

Fact 4. The map described above induces a bijection between the set $H^1(D^{(n)} \times T, \text{Hom}(V, E \otimes N_{D/S}^{-n}))$ and the set of isomorphism classes of pairs (\mathcal{E}, ϕ) consisting of a vector bundle \mathcal{E} on $D^{(n)} \times T$ and a framing $\phi : \mathcal{E}|_{D^{(n-1)} \times T} \rightarrow (1 \times f)^* E^{(n-1)}$.

Condition (1), together with Cohomology and Base Change, implies that the R -module $H^1(D^{(n)} \times \mathbf{A}_{n-1}, \text{Hom}(V, E \otimes N_{D/S}^{-n})) \cong H^1(D \times \mathbf{A}_{n-1}, \text{End}(E) \otimes N_{D/S}^{-n})$ is projective, hence free. For any affine \mathbf{A}_{n-1} -scheme $f : T = \text{Spec } R' \rightarrow \mathbf{A}_{n-1}$ we thus have

$$H^1(D^{(n)} \times T, \text{Hom}(V, E \otimes N_{D/S}^{-n})) = H^1(D^{(n)} \times \mathbf{A}_{n-1}, \text{Hom}(V, E \otimes N_{D/S}^{-n})) \otimes_R R'.$$

But this identity, combined with Fact 4, implies that \mathcal{A}_n is represented as a functor over \mathbf{A}_{n-1} by $\text{Spec Sym}^\bullet H^1(D^{(n)} \times \mathbf{A}_{n-1}, \text{Hom}(V, E \otimes N_{D/S}^{-n}))^*$, thus proving the proposition. \square

Proof of Theorem 1. There is a compatible family of morphisms $F_n : \mathrm{TF}_S(E) \rightarrow \mathbf{A}_n$ given by restriction. Fix a $\mathrm{Spec} \mathbf{C}$ -valued point of $\mathrm{TF}_S(E)$, that is, a point $u \in U$ together with an E_u -framed pair (\mathcal{F}, ϕ) on S . We will show that there is an open subfunctor Z of $\mathrm{TF}_S(E)$ that contains (\mathcal{F}, ϕ) and is represented by a scheme.

Fix a polarization H of S , and choose m sufficiently large that

1. $\mathcal{F} \otimes H^m$ is globally generated and
2. $H^1(\mathcal{F} \otimes H^m) = H^2(\mathcal{F} \otimes H^m) = 0$.

Further, fix n sufficiently large that the restriction map

$$H^0(\mathcal{F} \otimes H^m) \rightarrow H^0(\mathcal{F} \otimes H^m|_{D^{(n)}})$$

is injective; it is possible to choose such an n because \mathcal{F} is torsion-free. Finally, choose m' sufficiently large that $H^1(\mathcal{F} \otimes H^{m+m'}|_{D^{(n)}}) = 0$.

Next, let $Z \subseteq \mathrm{TF}_S(E)$ denote the open subfunctor parametrizing those triples $(W \xrightarrow{f} U, \mathcal{E}, \phi : \mathcal{E}|_{D \times W} \rightarrow (1 \times f)^*E)$ for which the family \mathcal{E} satisfies the following conditions:

- a. $\mathcal{E}_w \otimes H^m$ is globally generated for all $w \in W$,
- b. $H^1(\mathcal{E}_w \otimes H^m) = H^2(\mathcal{E}_w \otimes H^m) = 0$ for all $w \in W$,
- c. the map $H^0(\mathcal{E}_w \otimes H^m) \rightarrow H^0(\mathcal{E}_w \otimes H^m|_{D^{(n)}})$ is injective for all $w \in W$, and
- d. $H^1(\mathcal{E}_w \otimes H^{m+m'}|_{D^{(n)}}) = 0$ for all $w \in W$.

In the previous section we showed that there is a universal vector bundle $E^{(n)}$ on $D^{(n)} \times \mathbf{A}_n$. Fix an element of $Z(W)$; then the map $F_n(W) : W \rightarrow \mathbf{A}_n$ yields a vector bundle $(1 \times F_n)^*E^{(n)}$ on $D^{(n)} \times W$ together with an isomorphism

$$\mathcal{E}_W|_{D^{(n)} \times W} \xrightarrow{\phi_n} (1 \times F_n)^*E^{(n)};$$

here \mathcal{E}_W denotes the torsion-free sheaf on $S \times W$ determined by the fixed element of $Z(W)$. Let p_W denote the projection $S \times W \rightarrow W$. Then by construction the sheaves $(p_W)_*\mathcal{E}_W \otimes H^m$, $(p_W)_*\mathcal{E}_W \otimes H^{m+m'}$, and $(p_W)_*(\mathcal{E}_W \otimes H^{m+m'}|_{D^{(n)} \times W})$ are vector bundles on W , and, choosing a section s of $H^{m'}$ the zero locus of which has transverse intersection with D , there is a commutative diagram

$$\begin{array}{ccc} (p_W)_*\mathcal{E}_W \otimes H^m & \longrightarrow & (p_W)_*(\mathcal{E}_W \otimes H^m|_{D^{(n)} \times W}) \\ \downarrow \otimes s & & \downarrow \otimes s \\ (p_W)_*\mathcal{E}_W \otimes H^{m+m'} & \longrightarrow & (p_W)_*(\mathcal{E}_W \otimes H^{m+m'}|_{D^{(n)} \times W}) \end{array}$$

for which the vertical arrows (given by tensoring with s) and the top row are injective. Using ϕ_n , we may replace this diagram canonically with the diagram

$$\begin{array}{ccc} (p_W)_*(\mathcal{E}_W \otimes H^m) & \longrightarrow & (p_W)_*((1 \times F_n)^*E^{(n)} \otimes H^m) \\ \downarrow \otimes s & & \downarrow \otimes s \\ (p_W)_*\mathcal{E}_W \otimes H^{m+m'} & \longrightarrow & (p_W)_*((1 \times F_n)^*E^{(n)} \otimes H^{m+m'}). \end{array}$$

Now, by assumption (d) on W , we have

$$(p_W)_* \left((1 \times F_n)^* E^{(n)} \otimes H^{m+m'} \right) = F_n^* \left((p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \right),$$

where $p_{\mathbf{A}_n} : D^{(n)} \times \mathbf{A}_n \rightarrow \mathbf{A}_n$ is the projection, and so finally we obtain the diagram of vector bundles

$$\begin{array}{ccc} (p_W)_* \mathcal{E}_W \otimes H^m & & \\ \downarrow \otimes s & \searrow r & \\ (p_W)_* \mathcal{E}_W \otimes H^{m+m'} & \longrightarrow & F_n^* \left((p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \right) \end{array}$$

on W , where the diagonal map r and the map $\otimes s$ are injective. By construction, furthermore, the image of the morphism r is a vector subbundle of $F_n^* \left((p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \right)$ and consequently determines a morphism $W \rightarrow \mathbf{Gr}$ over \mathbf{A}_n , where $\mathbf{Gr} \xrightarrow{q} \mathbf{A}_n$ denotes the relative Grassmannian for the vector bundle $(p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'})$ on \mathbf{A}_n , the fiber of which over $a \in \mathbf{A}_n$ parametrizes vector subspaces of $H^0(E^{(n)} \otimes H^{m+m'})$ that are of dimension $h^0(\mathcal{F} \otimes H^m)$.

We now construct a Quot-scheme over \mathbf{Gr} that we will use to represent Z . We may pull back $(p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'})$ to \mathbf{Gr} to obtain a vector bundle $q^*(p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'})$ on (an open subset of) \mathbf{Gr} , with universal subbundle

$$\mathcal{U} \subset q^*(p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'})$$

of rank $h^0(\mathcal{F} \otimes H^m)$. If $p_{\mathbf{Gr}} : S \times \mathbf{Gr} \rightarrow \mathbf{Gr}$ denotes the projection to \mathbf{Gr} , we obtain a bundle $p_{\mathbf{Gr}}^* \mathcal{U} \subset p_{\mathbf{Gr}}^* q^*(p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'})$ on $S \times \mathbf{Gr}$, as well as a quotient

$$p_{\mathbf{Gr}}^* q^*(p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \rightarrow (1 \times q)^*(E^{(n)} \otimes H^{m+m'})$$

and subquotient $(1 \times q)^*(E^{(n)} \otimes H^m) \subset (1 \times q)^*(E^{(n)} \otimes H^{m+m'})$ that are sheaves on $S \times \mathbf{Gr}$ supported on $D^{(n)} \times \mathbf{Gr}$.

Consider the relative Quot-scheme $q' : \text{Quot}_{S \times \mathbf{Gr}/S}(p_{\mathbf{Gr}}^* \mathcal{U}) \rightarrow \mathbf{Gr}$ that parametrizes quotient sheaves for the family $p_{\mathbf{Gr}}^* \mathcal{U}$ on $S \times \mathbf{Gr}/S$. There is a universal quotient $(1 \times q')^* p_{\mathbf{Gr}}^* \mathcal{U} \rightarrow \mathcal{Q}$ on $S \times \text{Quot}_{S \times \mathbf{Gr}/S}$, giving a diagram

$$(2) \quad \begin{array}{ccc} (1 \times q')^* p_{\mathbf{Gr}}^* \mathcal{U} & \longrightarrow & (1 \times q')^* p_{\mathbf{Gr}}^* q^*(p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \\ \downarrow & & \downarrow \\ \mathcal{Q} & & (1 \times qq')^*(E^{(n)} \otimes H^m) \subset (1 \times qq')^*(E^{(n)} \otimes H^{m+m'}). \end{array}$$

There is a closed subscheme of $\text{Quot}_{S \times \mathbf{Gr}/S}$ (see the proof of Theorem 1.6 of [Ser86]) that represents the subfunctor of those quotients the kernels of which project to zero in $(1 \times qq')^*(E^{(n)} \otimes H^{m+m'})$, and a closed subscheme \mathcal{C} of that closed subscheme that represents the sub-subfunctor that parametrizes those quotients the images of which in $(1 \times qq')^*(E^{(n)} \otimes H^{m+m'})$ actually lie in the subsheaf $(1 \times qq')^*(E^{(n)} \otimes H^m)$. \mathcal{C} then represents the functor of quotients of $p_{\mathbf{Gr}}^* \mathcal{U}$ that map to $(1 \times qq')^*(E^{(n)} \otimes H^m)$ —that

is, it is exactly the closed subscheme over which Diagram (2) extends to

$$(3) \quad \begin{array}{ccc} (1 \times q')^* p_{\mathbf{Gr}}^* \mathcal{U} & \longrightarrow & (1 \times q')^* p_{\mathbf{Gr}}^* q^*(p_{\mathbf{A}_n})_*(E^{(n)} \otimes H^{m+m'}) \\ \downarrow & & \downarrow \\ \mathcal{Q} & \longrightarrow & (1 \times qq')^*(E^{(n)} \otimes H^m) \subset (1 \times qq')^*(E^{(n)} \otimes H^{m+m'}). \end{array}$$

Restricting further to an open subscheme \mathcal{C}° of \mathcal{C} , we may assume that, over \mathcal{C}° , the map $\mathcal{Q}|_{D^{(n)} \times \mathcal{C}^\circ} \rightarrow (1 \times qq')^*(E^{(n)} \otimes H^m)$ is an isomorphism, that \mathcal{Q} is a family of torsion-free sheaves on S , and that conditions (a) through (d) are satisfied.

By construction the morphism $W \rightarrow \mathbf{Gr}$ lifts to a morphism $W \rightarrow \mathcal{C}^\circ$; this construction thus determines a morphism of functors $Z \rightarrow \mathcal{C}^\circ$. Similarly, there is a forgetful morphism $\mathcal{C}^\circ \rightarrow Z$. Finally, it is clear from the construction that these two morphisms of functors are inverses of each other, as desired. \square

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