DEGREES OF CONVEX DEPENDENCE IN RECURSIVELY ENUMERABLE VECTOR SPACES

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Abstract. Let W be a recursively enumerable vector space over a recursive ordered field. We show the Turing equivalence of the following sets: the set of all tuples of vectors in W which are linearly dependent; the set of all tuples of vectors in W whose convex closures contain the zero vector; and the set of all pairs (X, Y) of tuples in W such that the convex closure of X intersects the convex closure of Y. We also form the analogous sets consisting of tuples with given numbers of elements, and prove similar results on the Turing equivalence of these.

1. Introduction

Early work which combined recursion theory and algebra had two sorts of goals. First, various techniques in recursion theory enabled the investigation of questions regarding the effectiveness of certain constructions in algebra. Second, some hoped to enrich algebra itself by the additional structure imposed on algebraic constructions by the notion of computability, for example the structure provided by recursive equivalence types. In [MN75] and [MN77], Metakides and Nerode pursued both sorts of questions. They showed, for example, that one cannot always extend a given recursive independent set to a basis for a recursive vector space; furthermore they introduced Turing degrees to the study of vector spaces and their linear dependence relations, and investigated the properties of those degrees.

In [Kal81], Kalantari blended these concerns with the notions of convexity and separation, and determined the effective content of the Separation Theorem of M.H. Stone. In [Dow84], Downey continued to investigate the lattice of r.e. convex subsets of a "fully effective" vector space, i.e. a recursive space which has recursive algorithms for determining linear dependence and convexity of tuples of vectors. In [Sho78], Shore examined the Turing degrees of the sets D_k, one for each $k \in \omega$, consisting of all k-tuples which are linearly dependent modulo the congruence relation of their vector space.

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Here we show that Kalantari's asumption of access to both a recursive dependence algorithm and a recursive convexity algorithm is redundant, since the two kinds of algorithms are Turing equivalent over any effectively presented vector space. In addition, we determine results on convexity sets analogous to Shore's results on degrees of linear dependence sets. Unlike most projects combining recursion theory and algebra, we derive all these results by algebraic rather than recursion-theoretic methods.

2. Subspace Geometry

In this section we derive those propositions of linear algebra which will be necessary for our results on convexity in later sections. Most of the results presented here are available in some form in the linear programming literature, but often in less than useable form; thus we state and prove them here in a style more suitable to our purpose.

We show first that if a subspace of a finite dimensional space intersects the positive orthant then its orthogonal complement fails to intersect the positive orthant.

Definition 2.1. Throughout this section, let F be an ordered field. Let F^n denote the n-dimensional vector space over the field F consisting of n-tuples of elements from F. Let $\mathbf{e_1}, \ldots, \mathbf{e_n}$ denote the standard ordered basis for F^n. If $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ are vectors in F^n, we let $\mathbf{v} \geqslant \mathbf{w}$ indicate that for all i such that $1 \leqslant i \leqslant n$, $v_i \geqslant w_i$. Let $(-|-)$ denote the standard inner product on F^n, i.e. $(\mathbf{x}|\mathbf{y}) = x_1y_1 + \cdots + x_ny_n$.

Definition 2.2. The *positive orthant* of F^n is the set of vectors

$$\{\mathbf{v} = (v_1, v_2, \ldots, v_n) \in F^n| \text{ for all } i \leqslant n,\, 0 < v_i \}.$$

The *positive orthant with boundary* of F^n is the set of vectors

$$\{\mathbf{v} = (v_1, v_2, \ldots, v_n) \in F^n| \text{ for all } i \leqslant n,\, 0 \leqslant v_i \} \setminus \{\mathbf{0}\}.$$

Proposition 2.3. If a subspace V of F^n intersects the positive orthant of F^n, then no vector in the positive orthant with boundary of F^n is orthogonal to every element of V.

Proof. Suppose V intersects the positive orthant in a vector $\mathbf{v} = (v_1, \ldots, v_n)$. Then for all i, $0 < v_i$. Now suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ is an element of the positive orthant with boundary of F^n. Then for all i, $0 \leqslant x_i$. Then the quantity $(\mathbf{v}|\mathbf{x}) = v_1x_1 + \cdots + v_nx_n$ is nonzero, since for all i, $v_i > 0$, and for some i we have $x_i > 0$. Thus $\mathbf{v}$ and $\mathbf{x}$ are not orthogonal. □

Next we show that if a subspace fails to intersect the positive orthant or its boundary, then the orthogonal complement of the subspace intersects the positive orthant.

Definition 2.4. Let L be a set of vectors of a finite-dimensional vector space F^n. The *dual cone* L^* of L is the set of vectors

$$L^* = \{\mathbf{a} \in F^n \mid \text{for all } \mathbf{y} \in L,\ (\mathbf{a}|\mathbf{y}) \leqslant 0\}.$$

Proposition 2.5. For any two subsets X and Y of F^n, if $X \subseteq Y$ then $Y^* \subseteq X^*$.

Proof. Suppose $\mathbf{a} \in Y^*$. Then for all vectors $\mathbf{y} \in Y$, $(\mathbf{a}|\mathbf{y}) \leqslant 0$; but since $X \subseteq Y$, we have $(\mathbf{a}|\mathbf{x}) \leqslant 0$ for all $\mathbf{x} \in X$. By the definition of X^*, then, $\mathbf{a} \in X^*$, and consequently $Y^* \subseteq X^*$. □

Theorem 2.6. Let W be a subspace of an n-dimensional vector space F^n, such that
(1) if $\mathbf{w} \in W$ and $(w_1, \dots, w_{n-1}) \geqslant \mathbf{0}$ then $w_n \leqslant 0$.
Then there exists a vector $\mathbf{u_0} \in F^{n-1}$ and a scalar $v_0 > 0$ such that
(A) $\mathbf{u_0} \geqslant \mathbf{0}$, and
(B) $((\mathbf{u_0}, v_0)|\mathbf{w}) \leqslant 0$ for all $\mathbf{w} \in W$.

Proof. Let $T = \{\mathbf{t} \in F^n \mid \mathbf{t} \leqslant \mathbf{w}$ for some $\mathbf{w} \in W\}$. Then $T^* = \{\boldsymbol{\alpha} \in F^n \mid$ for all $\mathbf{t} \in T$, $(\boldsymbol{\alpha}|\mathbf{t}) \leqslant 0\ \}$. We show first that if $\alpha \in T^*$, then $\boldsymbol{\alpha} \geqslant \mathbf{0}$. For suppose $\boldsymbol{\alpha} \ngeqslant \mathbf{0}$. If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, then for some j where $1 \leqslant j \leqslant n$, we have $\alpha_j < 0$. Since $(-\mathbf{e_j})$ is an element of T because $\mathbf{0} \in W$, $(\boldsymbol{\alpha}| - \mathbf{e_j}) > 0$ for that j, and thus $\boldsymbol{\alpha} \notin T^*$.

As a result we can write T^* as

$$T^* = \{\boldsymbol{\alpha} \geqslant 0 | (\boldsymbol{\alpha}|\mathbf{t}) \leqslant 0 \text{ for all } \mathbf{t} \in T\}.$$

Furthermore, since $W \subseteq T$ and $(\boldsymbol{\alpha}|\mathbf{t}) \leqslant (\boldsymbol{\alpha}|\mathbf{w}) \leqslant 0$ for some $\mathbf{w} \in W$ whenever $\mathbf{t} \in T$, we can write T^* as

$$T^* = \{\boldsymbol{\alpha} \geqslant 0 | (\boldsymbol{\alpha}|\mathbf{w}) \leqslant 0 \text{ for all } \mathbf{w} \in W\}.$$

Now let $H = \{\mathbf{h} | (h_1, \dots, h_{n-1}) \geqslant \mathbf{0}$ and $h_n > 0\}$. If $T^* \cap H \neq \emptyset$, then $\mathbf{h} \in (T^* \cap H)$ implies that for all $\mathbf{w} \in W$, $(\mathbf{h}|\mathbf{w}) \leqslant 0$, which, setting the desired vector $(\mathbf{u_0}, v_0)$ equal to $\mathbf{h}$, proves the theorem.

Suppose $T^* \cap H = \emptyset$; we derive a contradiction. Since $T^* \cap H = \emptyset$, setting $K = \{\mathbf{u} | (u_1, \dots, u_{n-1}) \geqslant \mathbf{0}$ and $u_n = 0\}$ we have $T^* \subseteq K$. Then by Proposition 2.5, $K^* \subseteq T^{**}$. Now consider the vector $\boldsymbol{\delta}$ defined by $(\delta_1, \dots, \delta_{n-1}) = \mathbf{0}$ and $\delta_n > 0$. Clearly $\boldsymbol{\delta} \in K^*$, so $\boldsymbol{\delta} \in T^{**}$. Consequently, by Lemma 2.9, $\boldsymbol{\delta} \in T$, so there exists some $\mathbf{w} \in W$ such that $\mathbf{w} \geqslant \boldsymbol{\delta}$, contradicting assumption (1). □

Definition 2.7. Let $L = \{\mathbf{l_1}, \dots, \mathbf{l_k}\}$ be a finite set of vectors of a finite-dimensional vector space V over an ordered field F. The *convex cone* $L^\angle$ *spanned by* L is the set

$$L^\angle = \{c_1\mathbf{l_1} + \cdots + c_k\mathbf{l_k} \mid \text{for all } i,\ 0 \leqslant c_i\}.$$

Proposition 2.8 (Farkas). If A is a finite set of vectors in F^n, then $A^{**} = A^\angle$.

Proof. See [GT56]. Note that the result holds when F is any ordered field. □

Lemma 2.9. $T = T^{**}$.

Proof. Let B be the set containing the vectors

(1) $\mathbf{b_1}, \dots \mathbf{b_k}$, where these form a basis for the subspace W;

(2) $(-\mathbf{b_1}), \ldots, (-\mathbf{b_k})$, where $\mathbf{b_1}, \ldots, \mathbf{b_k}$ are as in (1);
(3) $(-\mathbf{e_1}), \ldots, (-\mathbf{e_n})$, where $\mathbf{e_1}, \ldots, \mathbf{e_n}$ form the standard basis for F^n.

We show first that $T = B^{\angle}$.

Since W is a subspace and by definition of T, it is clear that $B^{\angle} \subseteq T$. Now suppose $\mathbf{t} \in T$. Then $\mathbf{t} \leqslant \mathbf{w}$ for some $\mathbf{w} \in W$, so $\mathbf{t}$ can be expressed as a sum of two vectors, $\mathbf{t} = \mathbf{w} + \mathbf{r}$, where $\mathbf{r} \leqslant \mathbf{0}$. Since $\mathbf{w}, \mathbf{r} \in B^{\angle}$, $\mathbf{t} \in B^{\angle}$.

Now, we show that $T^* = B^*$. By Proposition 2.5, since $B \subseteq T$ we have $T^* \subseteq B^*$. We show that $B^* \subseteq T^*$.

Suppose $\mathbf{q} \in B^*$. Then for all $\mathbf{b} \in B$, $(\mathbf{q}|\mathbf{b}) \leqslant 0$. But every element of T is a convex sum of elements of B, so if $\mathbf{t} \in T$, we have $(\mathbf{q}|\mathbf{t}) = (\mathbf{q}|\sum c_i \mathbf{d_i}) = \sum c_i (\mathbf{q}|\mathbf{d_i}) \leqslant 0$ where for all i, $\mathbf{d_i} \in B$. So $\mathbf{q} \in T^*$.

Since $T^* = B^*$, we have $T^{**} = B^{**}$, so $T^{**} = B^{**} = B^{\angle} = T$ by the result of Farkas. □

Corollary 2.10. Let W be a subspace of a vector space F^n. If W fails to intersect the positive orthant with boundary, then there exists an element of the positive orthant of F^n which is orthogonal to every element of W.

Proof. We prove the corollary by constructing the desired vector. Repeating the argument of the previous theorem, we obtain a set of nonzero vectors $\{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$, such that $\mathbf{x_i}$ is greater than zero in its i^{th} entry; for all $\mathbf{w} \in W$, $(\mathbf{x_i}|\mathbf{w}) \leqslant 0$; and for all i, $\mathbf{x_i} \geqslant \mathbf{0}$. Take the sum $\mathbf{z} = \mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_n}$. Then for all $\mathbf{w} \in W$, $(\mathbf{z}|\mathbf{w}) = (\mathbf{x_1}|\mathbf{w}) + (\mathbf{x_2}|\mathbf{w}) + \cdots + (\mathbf{x_n}|\mathbf{w}) \leqslant 0$. Furthermore, since W is a subspace, for all $\mathbf{w} \in W$ we get $(\mathbf{z}|(-\mathbf{w})) = -(\mathbf{z}|\mathbf{w}) \leqslant 0$. Consequently, for all $\mathbf{w} \in W$, $(\mathbf{z}|\mathbf{w}) = 0$; and since $\mathbf{z} > \mathbf{0}$, this proves the corollary. □

We note that the results of Corollary 2.10 and Proposition 2.3 combine to yield necessary and sufficient conditions for a subspace to intersect the positive orthant of F^n, results which we use in the rest of this paper in the construction of algorithms for linear and convex dependence.

3. R.E. Presented Spaces and Dependence Algorithms

In this section we show the equivalence of effective algorithms for determining linear dependence and convex dependence in vector spaces with effective presentations.

Definition 3.1. An *r.e. presented space* V over a countable recursive field F consists of

(1) an r.e. subset $|V|$ of ω,
(2) operations of vector addition and scalar multiplication which are partial recursive,
(3) an r.e. congruence relation $\equiv$ on V such that $V \mod \equiv$ is a vector space.

We begin by presenting some results of Metakides and Nerode on r.e. presented spaces.

Definition 3.2. Let V_∞ be the $\aleph_0$-dimensional vector space over a countable recursive field F consisting of all finitely nonzero ω-sequences of elements of F under pointwise operations. Let $\mathcal{L}(V_\infty)$ denote the lattice of r.e. subspaces of V_∞.

Proposition 3.3 (Metakides and Nerode). Every r.e. presented space is recursively isomorphic to a vector space of the form V_∞ mod W with $W \in \mathcal{L}(V_\infty)$. Every vector space of the form V_∞ mod W with W $\in \mathcal{L}(V_\infty)$ is r.e. presented.

Definition 3.4. An r.e. presented space V has a *dependence algorithm* if the set of all n-tuples, for all $n \in \omega$, of vectors in V which are linearly dependent is a recursive set.

Proposition 3.5 (Metakides and Nerode). An r.e. presented space V has a dependence algorithm iff it has an r.e. basis.

Next we define convex dependence and convexity algorithms and prove that an r.e. presented space has these if and only if it has a linear dependence algorithm.

Definition 3.6. An r.e. presented space V has a *convex dependence algorithm* if the set of all n-tuples $\langle \mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n} \rangle$, for all $n \in \omega$, of vectors in V such that there exist $c_1, \dots, c_n \in F$ with

(1) $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} \equiv 0$,
(2) $c_1 + c_2 + \cdots + c_n = 1$,
(3) for all i such that $1 \leqslant i \leqslant n$, $0 \leqslant c_i \leqslant 1$

is a recursive set.
If $W \subseteq V$, where $W = \{\mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w_n}\}$, we let W^{CV} denote the set of all linear combinations $c_1\mathbf{w_1} + c_2\mathbf{w_2} + \cdots + c_n\mathbf{w_n}$ of elements of W such that

(1) $c_1 + \cdots + c_n = 1$, and
(2) for all i such that $1 \leqslant i \leqslant n$, $0 \leqslant c_i \leqslant 1$.

Definition 3.7. Let $\langle \mathbf{v_1}, \dots, \mathbf{v_n} \rangle$ and $\langle \mathbf{w_1}, \dots, \mathbf{w_m} \rangle$ denote an n-tuple and an m-tuple of vectors in an r.e. presented space V. Then V has a *convexity algorithm* if the union of the sets of ordered pairs

(1) $(\{\mathbf{x}\}, \langle \mathbf{v_1}, \dots, \mathbf{v_n} \rangle)$ with $\mathbf{x} \in V$, and $\mathbf{x} \in \langle \mathbf{v_1}, \dots, \mathbf{v_n} \rangle^{CV}$; and
(2) $(\langle \mathbf{v_1}, \dots, \mathbf{v_n} \rangle, \langle \mathbf{w_1}, \dots, \mathbf{w_m} \rangle)$ with $\langle \mathbf{v_1}, \dots, \mathbf{v_n} \rangle^{CV} \cap \langle \mathbf{w_1}, \dots, \mathbf{w_m} \rangle^{CV} \neq \emptyset$

is a recursive set.

Hereafter we assume that all the vector spaces which we consider are equipped with ordered base fields, as required for the notions of convex dependence and convexity algorithms. Note that in the literature no such distinction is made between the terms "convexity" and "convex dependence"; Kalantari, who introduced its use in [Kal81], uses what we call a "convexity algorithm." Obviously, if V has a convexity algorithm, it has a convex dependence algorithm; we determine in the next section that, over any r.e. presented space over a recursive ordered field, convexity, convex dependence and linear dependence are Turing equivalent notions.

Theorem 3.8. An r.e. presented space V over a recursive ordered field F has a convexity algorithm iff it has a linear dependence algorithm.

Proof. Suppose V has a convexity algorithm; we can use it to determine whether $\mathbf{0} \in W^{CV}$ for any subset W of V, and thus, given an n-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle$, we can determine whether it is linearly dependent by the following method:

We list all sets S which contain, for every i, exactly one of $\mathbf{v_i}$, $(-\mathbf{v_i})$. It is clear that the n-tuple itself is linearly dependent iff each of the S is linearly dependent. For suppose that $\langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle$ is a linearly dependent tuple. Then there exist coefficients d_i such that $\sum d_i \mathbf{v_i} \equiv \mathbf{0}$. Now let $c_i = |d_i| / (\sum |d_i|)$, and let $\mathbf{w_i} \equiv \mathbf{v_i}$ if $d_i > 0$, and $\mathbf{w_i} \equiv (-\mathbf{v_i})$ otherwise. Then it is evident that $\sum c_i \mathbf{w_i} \equiv \mathbf{0}$ and that $0 \leqslant c_i \leqslant 1$ and $\sum c_i = 1$. Thus the n-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle$ is linearly dependent iff one of the sets S is convexly dependent, i.e. iff $\mathbf{0} \in S^{CV}$ for some S.

Now suppose we have a linear dependence algorithm for V; we show that we can determine whether an arbitrary pair consisting of an n-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle$ and an m-tuple $\langle \mathbf{w_1}, \ldots, \mathbf{w_m} \rangle$ of vectors in V is dependent in the sense of a convexity algorithm. We enumerate a basis $\alpha_1, \ldots, \alpha_k$ for the space by use of the linear dependence algorithm, until we have determined an expression for each of the $\mathbf{v_i}$ and $\mathbf{w_i}$ as a linear combination of basis elements. We then form the matrix of coefficients

$$M = \begin{bmatrix} [\mathbf{v_1}] & [\mathbf{v_2}] & \ldots & [\mathbf{v_n}] & [-\mathbf{w_1}] & [-\mathbf{w_2}] & \ldots & [-\mathbf{w_m}] \\ 1 & 1 & \ldots & 1 & -1 & -1 & \ldots & -1 \end{bmatrix} \tag{1}$$

where each $\mathbf{v_i}$ and each $\mathbf{w_i}$ is a column vector. This matrix has $(n+m)$ columns, and $(k+1)$ rows, where k is the number of basis vectors. We add the extra row of constants to ensure that any solution we find will have the sum of its first n terms equal to the sum of its last m terms, so that we are sure the solution can be "scaled" so that each set of terms sums to 1. We claim that exactly one of the following holds:

(1) the null space of M intersects the set of those vectors in the positive orthant with boundary of F^{n+m} which have nonzero entries in at least one of the first n coordinates and one of the last m coordinates, where the sum of each the vector's first n entries is equal to the sum of its last m entries;
(2) there is a vector $\mathbf{c} = (c_1, \ldots, c_{n+m})$ of the positive orthant with boundary of F^{n+m}, where either $c_i > 0$ whenever $1 \leqslant i \leqslant n$ or $c_i > 0$ whenever $n+1 \leqslant i \leqslant n+m$, such that $\mathbf{c}$ is orthogonal to every element of the null space of M.

For suppose (1) fails. By Corollary 2.10, if the null space of M fails to intersect the positive orthant with boundary, then the positive orthant contains a vector orthogonal to every element of the null space. Furthermore, by repeated application of Theorem 2.6 as in Corollary 2.10, if the null space of M intersects the positive orthant only in vectors of the form $(0,\ldots,0,c_1,\ldots,c_m)$, where $c_i \geqslant 0$ for $1 \leqslant i \leqslant m$, then there exists a vector of the form $(x_1,\ldots,x_n,y_1,\ldots,y_m)$, where $x_i > 0$ for all i such that $1 \leqslant i \leqslant n$ and $y_j \geqslant 0$ for all j such that $1 \leqslant j \leqslant m$, which is orthogonal to every element of the null space of M. Similarly, if the null space intersects the positive orthant with boundary only in vectors of the form $(c_1,\ldots c_n,0,\ldots,0)$, we can find a vector $(x_1,\ldots,x_n,y_1,\ldots,y_m)$, where $y_j > 0$ for any j such that $1 \leqslant j \leqslant m$, and $x_i \geqslant 0$ for any i such that $1 \leqslant i \leqslant n$, which is orthogonal to every element of the null space of M. Thus (2) holds. Conversely if (2) holds, we have $(\mathbf{c}|\mathbf{x}) > 0$ for every $\mathbf{x}$ satisfying the conditions in (1), so no such vector $\mathbf{x}$ is in the null space of M and (1) fails.

We enumerate vectors of the positive orthant with boundary of F^{n+m}, until we find one such that either (1) or (2) holds (note that to determine whether the vector is orthogonal to every element of the null space of M we need only determine whether it is orthogonal to every element of a basis for the null space). Since it is evident that the pair is dependent in the sense of a convexity algorithm iff (1) holds, finding one such vector is enough to determine whether the pair is dependent in the required sense. □

Corollary 3.9. An r.e. presented space V over a recursive ordered field F has a linear dependence algorithm iff it has a convex dependence algorithm.

Proof. Given a convexity algorithm, we can determine a convex dependence algorithm by determining whether, for a given n-tuple $\langle \mathbf{v_1},\ldots,\mathbf{v_n}\rangle$, it is the case that $\mathbf{0} \in \langle \mathbf{v_1},\ldots,\mathbf{v_n}\rangle^{CV}$. Furthermore, given a convex dependence algorithm, i.e. given a method of determining for an arbitrary n-tuple $\langle \mathbf{v_1},\ldots,\mathbf{v_n}\rangle$ whether $\mathbf{0} \in \langle \mathbf{v_1},\ldots,\mathbf{v_n}\rangle^{CV}$, we can use the method of Theorem 3.8 to determine whether a given n-tuple $\langle \mathbf{v_1},\ldots,\mathbf{v_n}\rangle$ is linearly dependent. The corollary follows immediately. □

4. Dependence Degree and Convexity Degree

In this section we generalize our results to dependence algorithms of any degree over any r.e. vector space of the form $V_\infty \mod V$. We also introduce some new notation which enables us to refer to the Turing degrees of the various dependence problems.

Definition 4.1. We let $D(V)$ denote the set of all n-tuples of vectors from V_∞ which are linearly dependent $\mod V$. We let $C(V)$ denote the set of all n-tuples of vectors from V_∞ which are convexly dependent (in the sense of a convex dependence algorithm) $\mod V$. By Gödel numbering we identify V_∞ with ω, and subsets of V_∞ with subsets of ω. Similarly we identify all finite sequences from V_∞ with ω, and so identify $D(V)$ and $C(V)$ with subsets of ω. We let $d(D(V))$, $d(C(V))$ denote the Turing degrees of the Gödel-coded sets $D(V)$, $C(V)$ respectively.

The next four propositions, which together imply both that $d(C(V)) = d(D(V))$ and that convexity degree and convex dependence degree are identical over the r.e. degrees, are easy relativizations of the proofs of theorems 3.8 and 3.9. We sketch the proofs here, showing those parts which are relativizations and leaving to the reader the task of filling in what remains unchanged.

Proposition 4.2. Let $V \in \mathcal{L}(V_\infty)$ be a subspace of V_∞ over a recursive ordered field F. Then $D(V)$ is recursive in $C(V)$.

Proof. The n-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle$ is linearly dependent $\mod V$ iff one of the sets S formed as in Proposition 3.8 is convexly dependent $\mod V$. We use $C(V)$ to check for linear dependence of the n-tuple by checking the sets S for convex dependence. □

The reader may have noted that, in fact, $D(V) \leqslant_{tt} C(V)$ (for a definition, see [Soa87]).

Proposition 4.3. Let V be as in Proposition 4.2. Then $C(V)$ is recursive in $D(V)$.

Proof. Suppose we are given a Gödel coding of $D(V)$. Given an n-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle$ of vectors in $V_\infty \mod V$, we can determine whether they are convexly dependent in the sense of a convex dependence algorithm as follows:

Since we have a dependence algorithm for $V_\infty \mod V$ which is recursive in $D(V)$, we can enumerate, recursively in $D(V)$, a basis for $V_\infty \mod V$ and determine each vector in our n-tuple as a linear combination $\mod V$ of finitely many elements of our basis. We form, for each vector $\mathbf{v_i}$ of the n-tuple (where $1 \leqslant i \leqslant n$), a vector $\mathbf{c_i}$ of the coefficients expressing $\mathbf{v_i}$ in terms of that basis. The vectors $\mathbf{c_i}$ are used to form the matrix

$$\left[\begin{bmatrix} \mathbf{c_1} \end{bmatrix} \; \begin{bmatrix} \mathbf{c_2} \end{bmatrix} \; \ldots \; \begin{bmatrix} \mathbf{c_n} \end{bmatrix} \right] \tag{2}$$

We then determine, as in Theorem 3.8, whether the null space of the matrix intersects the positive orthant with boundary of the appropriate vector space over F. □

Although as it stands the last reduction is not any kind of truth-table reduction, we will see in Proposition 5.4 that this reduction could be modified so that $C(V) \leqslant_{wtt} D(V)$.

Definition 4.4. Let $V \in \mathcal{L}(V_\infty)$ be a subspace of V_∞ over a recursive ordered field F. We let $Cv(V)$ denote the set of all ordered pairs consisting of an n-tuple $X = \langle \mathbf{x_1}, \ldots, \mathbf{x_n} \rangle$ and an m-tuple $Y = \langle \mathbf{y_1}, \ldots, \mathbf{y_m} \rangle$ of vectors in V_∞ such that there exists a vector $\mathbf{z}$ in V_∞ characterized by

$$\mathbf{z} \equiv_V \sum c_i \mathbf{x_i} \equiv_V \sum d_j \mathbf{y_j}$$

where, for all $1 \leqslant i \leqslant n$, $c_i \in F$, and for all $1 \leqslant j \leqslant m$, $d_j \in F$, and the following hold:

(1) for all i, $0 \leqslant c_i \leqslant 1$;
(2) $\sum c_i = 1$;
(3) for all j, $0 \leqslant d_j \leqslant 1$;
(4) $\sum d_j = 1$.

Note that this definition of $Cv(V)$ is the extension of the notion of a convexity algorithm to $V_\infty \mod V$ analogous to the extension of a convex dependence algorithm by means of $C(V)$.

Once again, we Gödel code the sets in $Cv(V)$, and denote the Turing degree of the coded version of $Cv(V)$ by $d(Cv(V))$.

Proposition 4.5. Let V be as in Proposition 4.2. Then $D(V)$ is recursive in $Cv(V)$.

Proof. Given the Gödel-coded form of $Cv(V)$, we decode and obtain the set of pairs (X, Y). In particular, when we decode we have every pair of the form $(X, \{\mathbf{0}\})$, where this implies that X is convexly dependent (in the sense of a convex dependence algorithm) $\mod V$. We check, as in Theorem 3.8, to see whether any of the sets S composed of, for every i such that $1 \leqslant i \leqslant n$, exactly one of $\mathbf{x_i}$, $(-\mathbf{x_i})$ is to be found in $Cv(V)$ in either the form $(S, \{\mathbf{0}\})$ or the form $(\{\mathbf{0}\}, S)$. Then the n-tuple $X = \langle \mathbf{x_1}, \dots, \mathbf{x_n} \rangle$ is linearly dependent $\mod V$ iff at least one of the pairs containing S and $\{\mathbf{0}\}$ for some S appears in $Cv(V)$. $\square$

Note that in fact $D(V) \leqslant_{tt} Cv(V)$ by the same reduction.

Proposition 4.6. Let V be as in Proposition 4.2. Then $Cv(V)$ is recursive in $D(V)$.

Proof. Suppose we are given the set $D(V)$; as before, we show how to determine the characteristic function of $Cv(V)$. Given a pair (X, Y) whose convexity we wish to test, we enumerate a basis for $V_\infty \mod V$ using our dependence algorithm $D(V)$, until we have enumerated enough vectors to enable us to express each vector $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ as a linear combination $\mod V$ of basis elements. Having thus found expressions $\mod V$ for all vectors in X and Y, we form, as in Proposition 3.8, the matrix of coefficients

$$\begin{bmatrix} [\mathbf{x_1}] & [\mathbf{x_2}] & \dots & [\mathbf{x_n}] & [-\mathbf{y_1}] & [-\mathbf{y_2}] & \dots & [-\mathbf{y_m}] \\ 1 & 1 & \dots & 1 & -1 & -1 & \dots & -1 \end{bmatrix} \tag{3}$$

with entries in F. We then search for an element in the positive orthant with boundary which lies either in the null space of the matrix or its orthogonal complement, according to the rules used in Theorem 3.8, and thus determine whether the pair (X, Y) belongs in $Cv(V)$. $\square$

Again we note that, as we will see in Proposition 5.4, this reduction could be modified so that $Cv(V) \leqslant_{wtt} D(V)$.

Finally we show, in Theorem 4.8, a variation of a result of Metakides and Nerode on linear dependence degrees which holds for convexity and convex dependence degrees as well: that we can find, for any r.e. degree $\mathbf{d}$, a space with convexity and convex dependence degrees $\mathbf{d}$.

Proposition 4.7 (Metakides and Nerode). Let V be a subspace of V_∞. Let $d(V)$ denote the Turing degree of the Gödel-coded version of the subset V of V_∞. Then

(1) $d(V) \leqslant d(D(V))$
(2) $d(D(V)) \leqslant d(V) \vee d(B)$ for B any basis for $V_\infty \mod V$

(if $\vee$ denotes the degree lattice-theoretic join).

Proof.

(1) Note that $\mathbf{v} \in V$ iff $\{\mathbf{v}\}^{CV} \in D(V)$, so that V is recursive in $D(V)$.
(2) By relativizing Proposition 3.5, we obtain the result that $D(V)$ is recursive in the join of B and V. □

Theorem 4.8. Let V_∞ be the $\aleph_0$-dimensional vector space over an infinite recursive ordered field F. Then there exists, for any r.e. degree $\mathbf{d}$, a $V \in \mathcal{L}(V_\infty)$ with convex dependence degree $d(C(V)) = \mathbf{d}$ and convexity degree $d(Cv(V)) = \mathbf{d}$.

Proof. Let $\gamma \subseteq \omega$ be an r.e. set of degree $\mathbf{d}$. Let $\mathbf{e_1}, \mathbf{e_2}, \ldots$ be the standard recursive basis of V_∞, and let W be the subspace generated by $\{\mathbf{e_i} | i \in \gamma\}$. By construction, $d(V) = \mathbf{d}$, and, since the basis $\{\mathbf{e_i} | i \notin \gamma\}$ for $V_\infty \mod W$ is recursive in γ, by the previous proposition $d(D(W)) = \mathbf{d}$; thus by Propositions 4.2, 4.3, 4.5, and 4.6, $d(C(W)) = d(Cv(W) = \mathbf{d}$. □

5. Controlling Dependence Degrees

In this section we examine the degrees of the $(n, m)^{th}$ convexity and k^{th} convex dependence sets. The proofs of the previous section leave the relationship between the degrees of the k^{th} dependence sets $D_k(V)$ and the analogous subsets of $C(V)$ and $Cv(V)$ unclear; we are forced in this section to develop some further minor results of linear algebra in order to determine the degrees of these sets.

Definition 5.1. Let $V \in \mathcal{L}(V_\infty)$ be a subsapce of V_∞. We define the k^{th} *linear dependence set* of V, denoted by $D_k(V)$, as the set of all k-tuples $\langle \mathbf{v_1}, \ldots, \mathbf{v_k} \rangle$ of vectors in V_∞, for fixed k, such that $\langle \mathbf{v_1}, \ldots, \mathbf{v_k} \rangle \in D(V)$.

It is clear that $D_i(V) \leqslant_T D_{i+1}(V)$ since, given an i-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_i} \rangle$ we can determine its membership in $D_i(V)$ as follows: we choose (i+1) many elements $\mathbf{e_p}$ which, as a set, are independent $\mod V$, and check the (i+1)-tuples $\langle \mathbf{v_1}, \ldots, \mathbf{v_i}, \mathbf{e_p} \rangle$, one tuple for each $\mathbf{e_p}$, for membership in $D_{i+1}(V)$. The i-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_i} \rangle$ is linearly dependent iff for all of the $\mathbf{e_p}$ the (i+1)-tuples $\langle \mathbf{v_1}, \ldots, \mathbf{v_i}, \mathbf{e_p} \rangle$ are linearly dependent. In [Sho78], Shore showed that the following result held:

Theorem 5.2 (Shore). Let $A_1, A_2, \ldots, A_0$ be a simultaneously r.e. sequence such that for $i > 0$, $A_i \leqslant_T A_{i+1}$ and $A_i \leqslant_T A_0$ uniformly. Then there is a $V \in \mathcal{L}(V_\infty)$ such that when $i > 0$, $D_i(V) \equiv_T A_i$ and $D(V) \equiv_T A_0$.

Definition 5.3. Let $V \in \mathcal{L}(V_\infty)$ be a subspace of V_∞. We define the k^{th} *convex dependence set* of V, denoted by $C_k(V)$, as the set of all k-tuples $\langle \mathbf{v_1}, \ldots, \mathbf{v_k} \rangle$ of vectors in V_∞, for fixed k, such that $\langle \mathbf{v_1}, \ldots, \mathbf{v_k} \rangle \in C(V)$. We define the $(n,m)^{th}$ *convexity set* of V, denoted by $Cv_{n,m}(V)$, as the set of all pairs (X, Y) consisting of an n-tuple X and an m-tuple Y of vectors in V_∞, for fixed n and m, such that $(X, Y) \in Cv(V)$.

We can show immediately that, for every i, $C_i(V) \equiv_T D_i(V)$.

Proposition 5.4. Let $V \in \mathcal{L}(V_\infty)$. Then for all $i > 0$, $C_i(V) \equiv_T D_i(V)$.

Proof. It is clear from our proof of Proposition 4.2 that we can determine the membership of i-tuples in $D_i(V)$ given the set $C_i(V)$, since the method we used to determine membership of an i-tuple in $D(V)$ used only information about similar i-tuples, for the same i, in $C(V)$. Thus we need only show that $C_i(V)$ is recursive in $D_i(V)$.

Given an i-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_i} \rangle$ of vectors in V_∞, we wish to determine whether it is an element of $C_i(V)$. Suppose the i-tuple $\langle \mathbf{v_1}, \ldots, \mathbf{v_i} \rangle$ is linearly independent mod V; we can determine this by checking for its membership in $D_i(V)$. Then the i-tuple is not a member of $C_i(V)$. Suppose, however, that the i-tuple is linearly dependent mod V. We find a basis B for the set of vectors by the following process:

We first find some independent (mod V) subset of the i-tuple, using $D_c(V)$ for any $c \leqslant i$, which we are allowed since each of these is recursive in $D_i(V)$. When we find such a subset S, we check to see whether, for every $\mathbf{v_k} \in \langle \mathbf{v_1}, \ldots, \mathbf{v_i} \rangle$, $S \cup \{\mathbf{v_k}\}$ is a dependent set. If so, since S is independent, S is a basis of the elements of the i-tuple. If not, then we take some $\mathbf{v_k}$ such that $S \cup \{\mathbf{v_k}\}$ is independent, add that $\mathbf{v_k}$ to S, and repeat the test. We will eventually produce a basis for the elements of the i-tuple.

When we have such a basis, we express each element of the i-tuple as a linear combination of elements of the basis, and proceed as in Proposition 4.3 to determine whether the null space of the coefficient matrix intersects the positive orthant of the appropriate space, where all the operations needed to determine this are operations in the recursive ordered field F. $\square$

The reader may note that the method used in Proposition 5.4 of finding a basis sufficient to express a given set of vectors could be used in Propositions 4.3 and 4.6 to limit the use of the characteristic function of $D(V)$ in a way that could be recursively determined, thus giving us weak truth-table reductions in those propositions. The reader may also note that by the reduction in Proposition 5.4 we have $D_i(V) \leqslant_{btt} C_i(V)$ with norm 2^i, and $C_i(V) \leqslant_{wtt} D_i(V)$.

Corollary 5.5. Let $A_1, A_2, \ldots, A_0$ be a simultaneously r.e. sequence such that for all $i > 0$, $A_i \leqslant_T A_{i+1}$ and $A_i \leqslant_T A_0$ uniformly. Then there is a $V \in \mathcal{L}(V_\infty)$ such that for all $i > 0$, $C_i(V) \equiv_T A_i$ and $C(V) \equiv_T A_0$.

Proof. Follows from Proposition 5.4 and Theorem 5.2. □

We show next that in general $Cv_{n,1} \leqslant_T D_n(V)$, after first proving a necessary result of linear algebra.

Lemma 5.6. Let W be an r.e. presented space. Let $Y = \{\mathbf{y_1}, \ldots, \mathbf{y_n}\}$ be a linearly dependent subset of W, and $\mathbf{x} \in W$ a vector. Suppose, for all i where $1 \leqslant i \leqslant n$, there exist $\lambda_i \geqslant 0$ such that $\mathbf{x} \equiv \lambda_1 \mathbf{y_1} + \cdots + \lambda_n \mathbf{y_n}$, and $\lambda_1 + \cdots + \lambda_n \leqslant 1$. Then there exists a linearly independent subset $Z = \{\mathbf{z_1}, \ldots, \mathbf{z_k}\}$ of Y such that for all j where $1 \leqslant j \leqslant k$, there exist $\sigma_j \geqslant 0$ such that $x \equiv \sigma_1 \mathbf{z_1} + \cdots + \sigma_k \mathbf{z_k}$ and $\sigma_1 + \cdots + \sigma_k \leqslant 1$.

Proof. We prove that, given such a linearly dependent set Y, the vector $\mathbf{x}$ can be expressed in terms of $n-1$ of the elements of Y with similar restrictions on the λ_i. The result follows by induction on the rank of the set Y.

Suppose, for some set of scalars c_i where $1 \leqslant i \leqslant n$, we have $c_1 \mathbf{y_1} + \cdots + c_n \mathbf{y_n} \equiv \mathbf{0}$. We show by cases that there exists an expression for $\mathbf{x}$ in terms of $n-1$ of the elements of Y:

Case 1. Suppose that for all i, we have $c_i \geqslant 0$. (Note that if for all i we have $c_i \leqslant 0$, we simply multiply each c_i by (-1).) For each $c_i \neq 0$, we find $\lambda_i c_i^{-1}$. When we have determined all such quotients, we take the least such quotient q, and form a new set of coefficients d_i for $1 \leqslant i \leqslant n$ where $d_i = qc_i$. Then for all i the coefficients $\lambda_i - d_i$ satisfy $0 \leqslant \lambda_i - d_i \leqslant 1$ and $\sum \lambda_i - d_i \leqslant 1$, and for some i we have $\lambda_i - d_i = 0$, thus satisfying the restrictions.

Case 2. Suppose that for some i and some j such that $1 \leqslant i, j \leqslant n$ we have $c_i > 0$ and $c_j < 0$. If $\sum c_i > 0$, we multiply each c_i by (-1); otherwise we do nothing. As before, for each nonzero c_i we find $\lambda_i c_i^{-1}$, and, choosing q to be the greatest such which is less than zero, we form the new coefficients $d_i = qc_i$. Then $\sum d_i > 0$, so the coefficients $\lambda_i - d_i$ satisfy the necessary restrictions, and at least one of them is equal to zero. □

Theorem 5.7. Let $V \in \mathcal{L}(V_\infty)$. Then for all $i > 0$, $Cv_{n,1} \leqslant_T D_n(V)$.

Proof. Suppose we are given a pair $(X, \{\mathbf{y}\})$ where X is an n-tuple of vectors in V_∞, and we wish to determine whether $(X, \{\mathbf{y}\}) \in Cv_{n,1}(V)$; i.e., if $X = \langle \mathbf{x_1}, \ldots, \mathbf{x_n} \rangle$, we wish to determine whether the equation

$$\lambda_1 \mathbf{x_1} + \cdots + \lambda_{n-1} \mathbf{x_{n-1}} + (1 - \sum \lambda_i) \mathbf{x_n} \equiv \mathbf{y}$$

has a solution such that for all i where $1 \leqslant i \leqslant n-1$, we have $0 \leqslant \lambda_i \leqslant 1$ and $\sum \lambda_i \leqslant 1$. Alternatively we can write this equation as

$$\lambda_1(\mathbf{x_1} - \mathbf{x_n}) + \lambda_2(\mathbf{x_2} - \mathbf{x_n}) + \cdots + \lambda_{n-1}(\mathbf{x_{n-1}} - \mathbf{x_n}) \equiv \mathbf{y} - \mathbf{x_n},$$

with the same conditions applying to the λ_i. It is obvious that, if this equation has a solution which uses only some linearly independent subset of $\{(\mathbf{x_1} - \mathbf{x_n}), \ldots, (\mathbf{x_{n-1}} - \mathbf{x_n})\}$, then the equation has a solution; so by the previous proposition the equation has a solution iff it has a solution using only a linearly independent subset of $\{(\mathbf{x_1} - \mathbf{x_n}), \ldots, (\mathbf{x_{n-1}} - \mathbf{x_n})\}$. Thus to determine whether $(X, \{\mathbf{y}\}) \in Cv_{n,1}$, we

(1) find all linearly independent subsets S of $\{(\mathbf{x_1} - \mathbf{x_n}), \ldots, (\mathbf{x_{n-1}} - \mathbf{x_n})\}$;
(2) use the information provided by $D_n(V)$ to determine whether $(\mathbf{y} - \mathbf{x_n})$ is dependent on each set S;
(3) enumerate linear combinations of elements of each set S until we find an expression for $(\mathbf{y} - \mathbf{x_n})$ in terms of each set S on which it is dependent;
(4) check to see whether the coefficients in the expression fit the requirements.

Note that for each such S the expression in (3) for $(\mathbf{y} - \mathbf{x_n})$ is unique, so we need only check that one expression which we have found. Then $(X, \{\mathbf{y}\}) \in Cv_{n,1}(V)$ iff one of these expressions satisfies the given restrictions on the λ_i. □

We note that $Cv_{n,1}(V) \leqslant_{wtt} D_n(V)$.

Corollary 5.8. Let $V \in \mathcal{L}(V_\infty)$. Then $Cv_{n,1}(V) \equiv_T D_n(V)$.

Proof. By checking whether $(X, \{\mathbf{0}\}) \in Cv_{n,1}(V)$ we can determine whether $X \in C_n(V)$, which is Turing equivalent to $D_n(V)$; the corollary follows from this and Theorem 5.7. □

Note that $D_n(V) \leqslant_{btt} Cv_{n,1}(V)$. We show finally that, for $0 < j < n$, we have $Cv_{(n-j),j}(V) \equiv_T D_n(V)$, thus completing our characterization of the dependence degrees.

Lemma 5.9. Let $A = \{\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_n}\}$ and $B = \{\boldsymbol{\beta_1}, \ldots, \boldsymbol{\beta_m}\}$ be sets of vectors in an r.e. presented space W over an ordered field F; let $\mathbf{y}$ be a vector in W. Let $\boldsymbol{\gamma_{ij}} = \boldsymbol{\alpha_i} + \boldsymbol{\beta_j}$ where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Then there exist $c_i \geqslant 0$ and $d_j \geqslant 0$ in F, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, such that

$$\sum_{i=1}^{n} c_i \boldsymbol{\alpha_i} + \sum_{j=1}^{m} d_j \boldsymbol{\beta_j} \equiv \mathbf{y}, \tag{4}$$

where $\sum c_i \leqslant 1$, and $\sum d_j \leqslant 1$, iff there exist $f_i \geqslant 0, g_j \geqslant 0$, and $h_{ij} \geqslant 0$ in F, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, such that

$$\sum_{i=1}^{n} f_i \boldsymbol{\alpha_i} + \sum_{j=1}^{m} g_j \boldsymbol{\beta_j} + \sum_{i=1}^{n} \sum_{j=1}^{m} h_{ij} \boldsymbol{\gamma_{ij}} \equiv \mathbf{y}, \tag{5}$$

where $\sum(f_i + g_j + h_{ij}) \leqslant 1$.

Proof. We first show that, given a solution to Equation 4 satisfying its associated conditions, we can construct a solution to Equation 5 satisfying its associated conditions.

Given a set of c_i and d_j satisfying Equation 4, we set $f_i = c_i$, $g_j = d_j$, and $h_{ij} = 0$. We choose some $f_i \neq 0$ and some $g_j \neq 0$, and subtract $\min\{f_i, g_j\}$ from each, adding at the same time $\min\{f_i, g_j\}$ to h_{ij}. We repeat the process until either $\sum f_i = 0$ or $\sum g_j = 0$. After each such repetition, the new set of coefficients f_i, g_j and h_{ij} remains a solution to Equation 4 if the previous set was a solution; furthermore, we add to h_{ij} exactly what we subtract from each of f_i and g_j, so after all repetitions are completed we have

$$\sum(f_i + g_j + h_{ij}) \leqslant 1.$$

Next we construct a solution to Equation 4 satisying its associated conditions from a set of f_i, g_j, h_{ij} satisfying Equation 5 and its associated conditions.

Given a set of f_i, g_j, h_{ij} satisfying Equation 5, we set

$$c_i = f_i + \sum_{j=1}^{m} h_{ij},$$

and set

$$d_j = g_j + \sum_{i=1}^{n} h_{ij}.$$

Then clearly the set of all c_i and d_j satisfies Equation 4. In addition, we have

$$\sum_{i=1}^{n} c_i = \left(\sum_{i=1}^{n} f_i\right) + \left(\sum_{i=1}^{n}\sum_{j=1}^{m} h_{ij}\right) \leqslant \sum(f_i + g_j + h_{ij}) \leqslant 1,$$

and

$$\sum_{j=1}^{m} d_j = \left(\sum_{j=1}^{m} g_j\right) + \left(\sum_{j=1}^{m}\sum_{i=1}^{n} h_{ij}\right) \leqslant \sum(f_i + g_j + h_{ij}) \leqslant 1.$$

Thus $\sum c_i \leqslant 1$ and $\sum d_j \leqslant 1$. □

Theorem 5.10. Let $V \in \mathcal{L}(V_\infty)$. Then for all $n > 0$ and $0 < j < n$, $Cv_{(n-j),j}(V) \equiv_T D_{n-1}(V)$.

Proof. First, we show that $C_{n-1}(V) \leqslant_T Cv_{(n-j),j}(V)$.

Suppose we have an (n-1)-tuple $T = \langle \mathbf{t_1}, \ldots, \mathbf{t_{n-1}} \rangle$ for whose membership in $C_{n-1}(V)$ we wish to test. We form all pairs (X, Y), consisting of an (n-j)-tuple X and a j-tuple Y, such that $X\hat{}(-Y)$, the concatenation of X and $(-Y)$, is an order permutation of T, where $(-Y)$ denotes the tuple formed by multiplying each vector of Y by (-1). We claim that $T \in C_{n-1}(V)$ iff $(X, Y\hat{}\mathbf{0}) \in Cv_{(n-j),j}(V)$ for some one of the pairs (X, Y). We can see this as follows: a tuple T is in $C_{n-1}(V)$ iff it is linearly dependent with coefficients between zero and one, the sum of which is one. Clearly if (X, Y) is a pair formed from the tuple T as above, and $(X, Y^\frown\mathbf{0}) \in Cv_{(n-j),j}(V)$, then the tuple is in $C_{n-1}(V)$.

Conversely, if the tuple is in $C_{n-1}(V)$, there is some set of coefficients $c_1, \ldots, c_{n-1}$ such that $c_1\mathbf{t_1} + \cdots + c_{n-1}\mathbf{t_{n-1}} \equiv \mathbf{0}$. We list the vectors in order so that $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{n-1}$, and let $X = \langle \mathbf{t_1}, \ldots, \mathbf{t_{n-j}} \rangle$, and $Y = \langle \mathbf{t_{n-j+1}}, \ldots, \mathbf{t_{n-1}} \rangle$. Then then pair $(X, (-Y)^\frown\mathbf{0})$ is in $Cv_{(n-j),j}(V)$, with coefficients given by

$$d_k = \frac{c_k}{\sum\limits_{i=1}^{n-j} c_i}$$

for $1 \leqslant k \leqslant n-1$, and

$$d_n = 1 - \sum_{i=n-j+1}^{n-1} c_i.$$

Thus $C_{n-1}(V) \leqslant_T Cv_{(n-j),j}(V)$, so by Proposition 5.4, $D_{n-1}(V) \leqslant_T Cv_{(n-j),j}(V)$.

Next we show that $Cv_{(n-j),j}(V) \leqslant_T D_{n-1}(V)$.

Suppose we have a pair $(X, (-Y))$ where $X = \langle \mathbf{x_1}, \ldots, \mathbf{x_{n-j}} \rangle$ and $Y = \langle \mathbf{y_1}, \ldots, \mathbf{y_j} \rangle$. We wish to determine whether

$$c_1\mathbf{x_1} + \cdots + c_{n-j}\mathbf{x_{n-j}} + d_1\mathbf{y_1} + \cdots + d_j\mathbf{y_j} \equiv \mathbf{0}$$

for some set of c_i, d_m where $1 \leqslant i \leqslant (n-j)$ and $1 \leqslant m \leqslant j$ such that $c_i \geqslant 0$, $d_m \geqslant 0$, and $\sum c_i = \sum d_m = 1$. Rewriting, we get

$$c_1\mathbf{x_1} + \cdots + c_{n-j-1}\mathbf{x_{n-j-1}} + \left(1 - \sum_{i=1}^{n-j-1} c_i\right)\mathbf{x_{n-j}} + d_1\mathbf{y_1} + \cdots + d_{j-1}\mathbf{y_{j-1}} + \left(1 - \sum_{m=1}^{j-1} d_m\right)\mathbf{y_j} \equiv \mathbf{0}$$

with $\sum_{i=1}^{n-j-1} c_i \leqslant 1$ and $\sum_{m=1}^{j-1} d_m \leqslant 1$, or

$$\sum_{i=1}^{n-j-1} c_i(\mathbf{x_i} - \mathbf{x_{n-j}}) + \sum_{m=1}^{j-1} d_m(\mathbf{y_m} - \mathbf{y_j}) \equiv (-\mathbf{x_{n-j}} - \mathbf{y_j}) \tag{6}$$

with $\sum_{i=1}^{n-j-1} c_i \leqslant 1$ and $\sum_{m=1}^{j-1} d_m \leqslant 1$. For each i let $\boldsymbol{\alpha_i} \equiv \mathbf{x_i} - \mathbf{x_{n-j}}$, and for each m let $\boldsymbol{\beta_m} \equiv \mathbf{y_m} - \mathbf{y_j}$. By Lemma 5.9, if we let $\boldsymbol{\gamma_{im}} \equiv \boldsymbol{\alpha_i} + \boldsymbol{\beta_m}$ for all i and m, then Equation 6 has a solution iff there is some solution to

$$\sum_{i=1}^{n-j-1} c_i\boldsymbol{\alpha_i} + \sum_{m=1}^{j-1} d_m\boldsymbol{\beta_m} + \sum_{i=1}^{n-j-1}\sum_{m=1}^{j-1} f_{im}\boldsymbol{\gamma_{im}} \equiv (-\mathbf{x_{n-j}} - \mathbf{y_j}) \tag{7}$$

with $0 \leqslant c_i, d_m, f_{im} \leqslant 1$ and $\sum(c_i + d_m + f_{im}) \leqslant 1$. By Lemma 5.6, this has a solution iff there is a solution to the analogous equation mentioning only a linearly independent subset of the set M containing all of the $\boldsymbol{\alpha_i}, \boldsymbol{\beta_m}$ and $\boldsymbol{\gamma_{im}}$. Since all these vectors lie in the space spanned by the $\boldsymbol{\alpha_i}$ and $\boldsymbol{\beta_m}$, any linearly independent subset of them contains at most $n-2$ vectors, so we can find all such linearly independent sets by using $D_{n-1}(V)$. We determine whether $(-\mathbf{x_{n-j}} - \mathbf{y_j})$ is linearly dependent on any of the linearly independent subsets of M; if so, we enumerate solutions to all such, as in Theorem 5.7. Then the pair $(X, (-Y)) \in Cv_{(n-j),j}(V)$ iff at least one of these expressions has coefficients satisfying the given conditions. □

Again we have $D_{n-1}(V) \leqslant_{btt} Cv_{(n-j),j}(V)$ and $Cv_{(n-j),j}(V) \leqslant_{wtt} D_{n-1}(V)$.

Corollary 5.11. Let $V \in \mathcal{L}(V_\infty)$. Let $0 < j \leqslant n$. Then

$$D_n(V) \equiv_T C_n(V) \equiv_T Cv_{(n-j+1),j}(V).$$

Proof. By Proposition 5.4 and Theorem 5.10. □

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