

## Math 525: Problem Set 9<sup>1</sup>

**Important note:** I will be away the week of November 16. On Monday and Wednesday, Tom Nevins will be filling in for me, and there will be no class on Friday, Nov 20. I will be available by email throughout the week to answer questions about this assignment.

**Due date:** In my mailbox in Altgeld 250 by noon on *Friday*, November 20.

1. Suppose  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a smooth map whose derivative  $Df$  at  $x$  is non-singular. As a matrix,  $Df$  is just the Jacobian  $(\frac{\partial f_i}{\partial x_j}(x))$ . You will show that the local degree is determined by  $Df$  via

$$\deg_x f = \epsilon(x) := \begin{cases} 1 & Df: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ preserves orientation, i.e. } \det Df > 0. \\ -1 & Df: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ reverses orientation, i.e. } \det Df < 0. \end{cases}$$

Set  $y = f(x)$ , and let  $V$  be any neighborhood of  $y$ . By the Inverse Function Theorem, there is a neighborhood  $U$  of  $x$  so that  $f$  restricted to  $U$  is a homeomorphism and  $f(U) \subset V$ .

- (a) Regard  $\mathbb{R}^n$  as  $S^n - \{\text{pt}\}$ . If we fix a generator  $\alpha \in H_n(S^n)$ , show this determines generators  $\bar{\alpha}$  of  $H_n(U, U - x)$  and  $\bar{\alpha}'$  of  $H_n(V, V - y)$ . Then define  $\deg_x f$  by  $f_*(\bar{\alpha}) = (\deg_x f)\bar{\alpha}'$ . Show that this does *not* depend on the choice of  $\alpha$ .
  - (b) Prove that  $\deg_x f$  does not depend on the choice of  $U$  and  $V$ .
  - (c) Consider the 1<sup>st</sup>-order Taylor approximation  $a(v) = y + (Df)(v - x)$  to  $f$  at  $x$ . Show that  $\deg_x f = \deg_0 Df$  by arguing that for some  $U$  the maps  $a, f: (U, U - x) \rightarrow (\mathbb{R}^n, \mathbb{R}^n - y)$  are homotopic, *as maps of pairs* (see page 36 of Hatcher for the definition).
  - (d) If  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an invertible linear map, show that  $\deg_0 T$  is 1 or  $-1$  depending on the sign of  $\det T$ . Hint: Use Gaussian elimination to show that the matrix  $T$  can be joined by a path of invertible matrices to a diagonal matrix with  $\pm 1$ 's on the diagonal.
2. A polynomial  $f(z) \in \mathbb{C}[z]$ , viewed as a map  $\mathbb{C} \rightarrow \mathbb{C}$ , always extends to a continuous map  $\hat{f}$  of the one-point compactification  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong S^2$ .
- (a) Show that  $\deg \hat{f}$  as a self-map of  $S^2$  is the same as the degree of  $f$  as a polynomial. One approach is to calculate  $\deg_\infty \hat{f}$ .
  - (b) Use (a) to prove the Fundamental Theorem of Algebra.
3. Construct a surjective map  $S^n \rightarrow S^n$  of degree zero, for each  $n \geq 1$ .
4. Hatcher, Section 2.2, #11.
5. Hatcher, Section 2.2, #13.
6. Hatcher, Section 2.2, #21.

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<sup>1</sup>Revised Nov. 13 to remove duplicate problems.