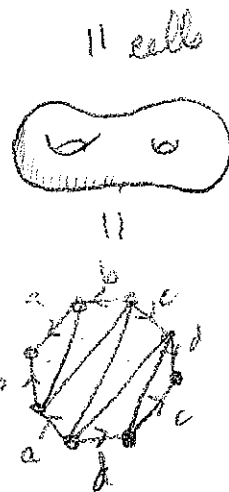
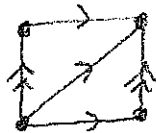
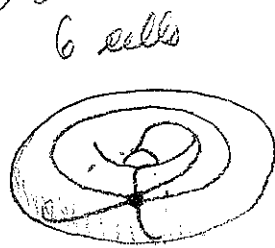


Lecture 32: Cellular Homology

Have simplicial homology of a Δ -complex, but still gets complicated.

A simplification is



Cellular Homology: X a CW complex,

$$f_\alpha: D^{n_\alpha} \rightarrow X \quad , \quad f'_\alpha: \partial D^{n_\alpha} \rightarrow X^{n_\alpha-1}$$

image a cell attaching map.

Define

$$C_n^{CW}(X) = H_n(X^n, X^{n-1}) \cong \tilde{H}_n(X^n / X^{n-1} \cong \bigvee_\alpha S^n)$$

$$= \bigoplus_\alpha \mathbb{Z} \quad \text{where } \alpha \text{ indexes the } n\text{-cells.}$$

The boundary map $d_n: C_n^{CW}(X) \rightarrow C_{n-1}^{CW}(X)$

is $d_n = j_* \circ \partial$

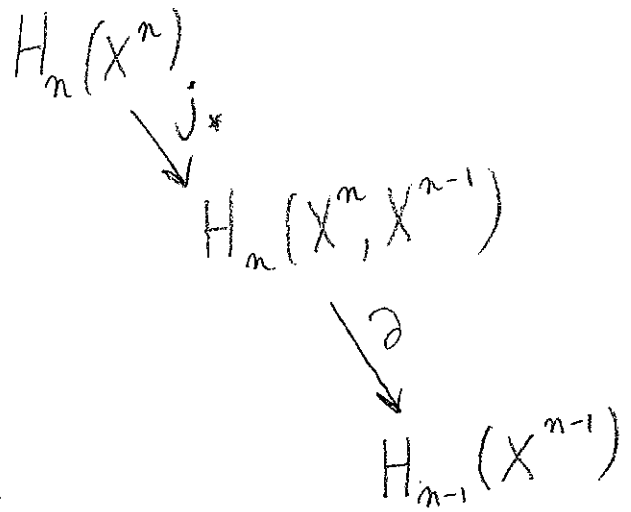
$$\begin{array}{ccccc}
 & \partial \rightarrow & H_n(X^n) & & \\
 & & \searrow j_* & & \\
 H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & \searrow \partial & & \nearrow j_* \\
 & & H_{n-1}(X^{n-1}) & &
 \end{array}$$

where ∂, j_* are all

from the long exact sequence of various pairs.

$$H_{n-1}(X^{n-1})$$

Note: $d_n \circ d_{n+1} = 0$ since
 is from the long exact seq.
 of the pair, hence exact.
 $\Rightarrow \partial \circ j_* = 0.$



Set $H_n^{CW}(X) = \ker d_n / \text{im } d_{n+1}.$

Thm: $H_*^{CW}(X) \cong H_*(X)$ ← usual singular homology.

Pf: See Hatcher. Uses prob 3 (= Hatcher #22) from
 takehome midterm #2. ▣

Geometric meaning of d : For $n > 1$, consider the
 gen $i_n \in H_n(D^n, \partial D^n)$. Given a cell map $f_\alpha: D^n \rightarrow X$,
 take $e_\alpha = (f_\alpha)_*(i_n) \in C_n^{CW}(X) = \bigoplus_\alpha \mathbb{Z}$, which
 generates the α^{th} factor. Now

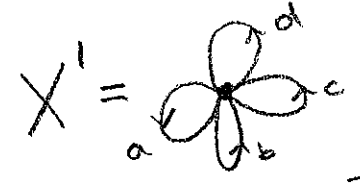
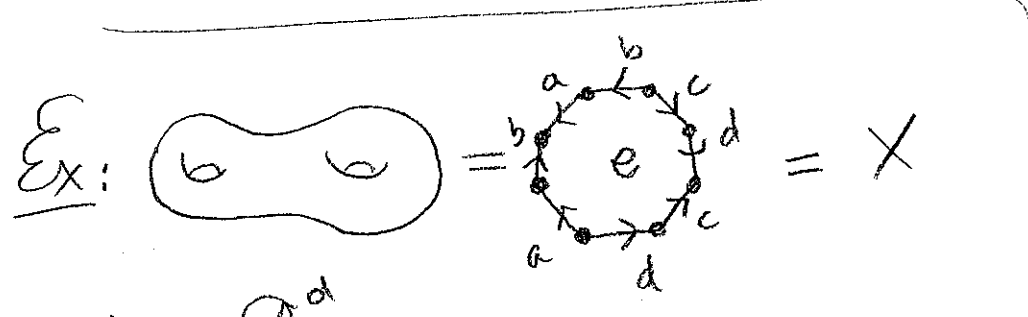
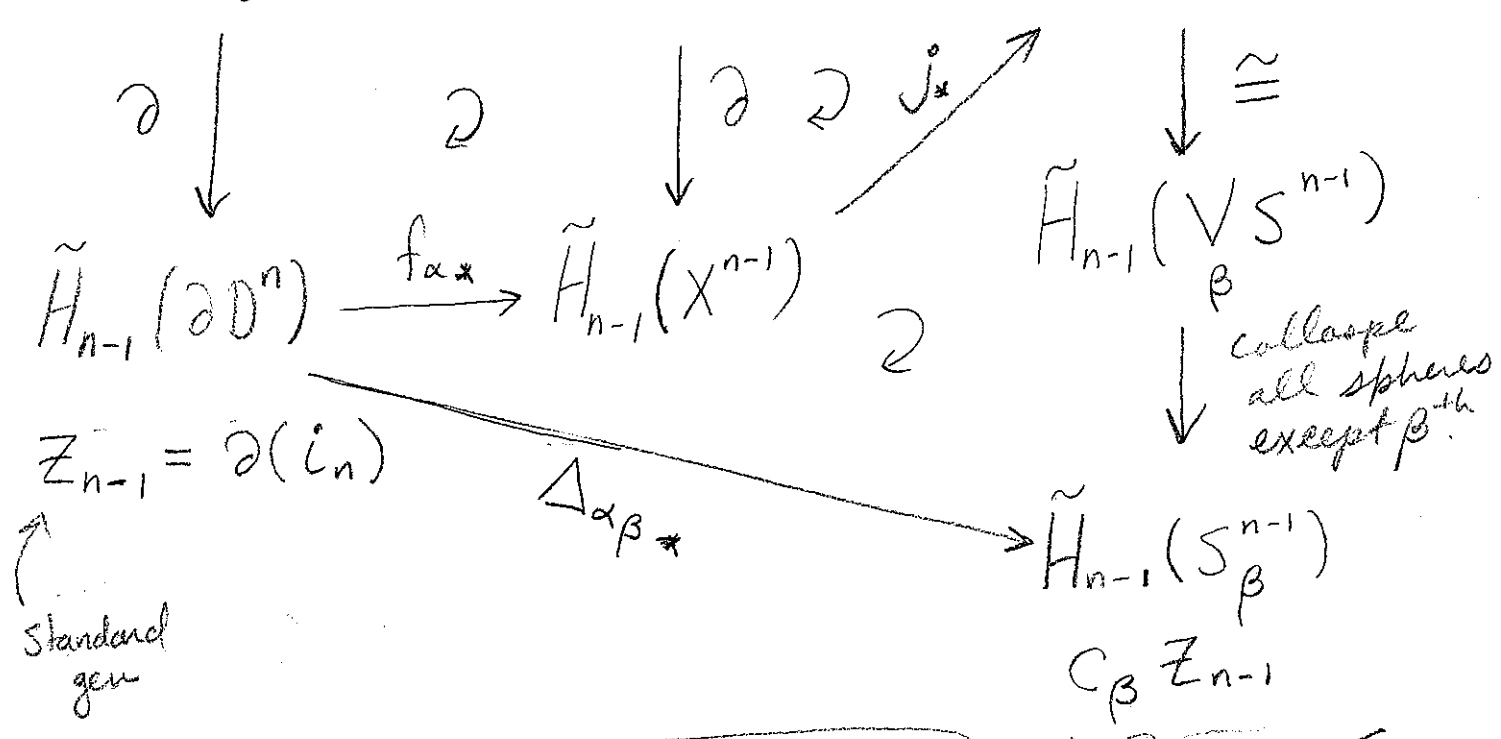
$$d_n(\alpha) = \sum_\beta c_\beta e_\beta \quad \text{where } \beta \text{ indexes the } (n-1)\text{-cells.}$$

Idea: $c_\beta = \deg \left(\partial D^n \xrightarrow{\Delta_{\alpha\beta}} \underbrace{X^{n-1} / X^{n-2}}_{Y = S_\beta^{n-1}} \right)$
 all $(n-1)$ -cells except β

where $\Delta_{\alpha\beta} = g \circ f_\alpha$
 \uparrow attaching map $\partial D^n \rightarrow X^{n-1}$
 \uparrow quotient map.

Reason: $i_n \xrightarrow{\quad} e_\alpha \xrightarrow{\quad} \sum c_\beta e_\beta$ (85)

$$H_n(D^n, \partial D^n) \xrightarrow{f_{\alpha*}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^n, X^{n-2})$$



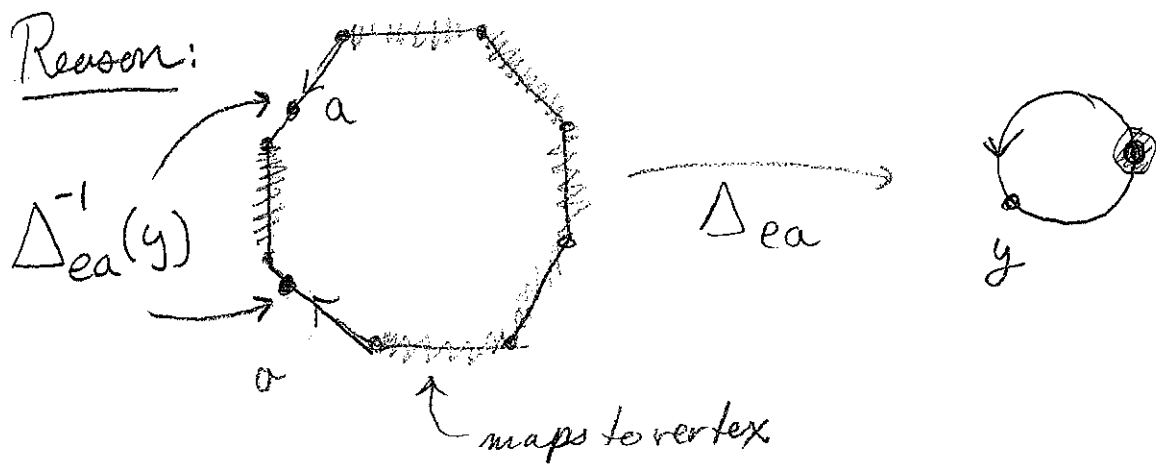
$$0 \rightarrow C_2^{CW}(X) \xrightarrow{\cong} C_1^{CW}(X) \xrightarrow{\cong} C_0^{CW}(X) \rightarrow 0$$

$d_2 = 0$ because the comp of $d_2(e)$ along a is

$d_1 = 0$; d_1 works the same as the usual simplicial ∂ map.

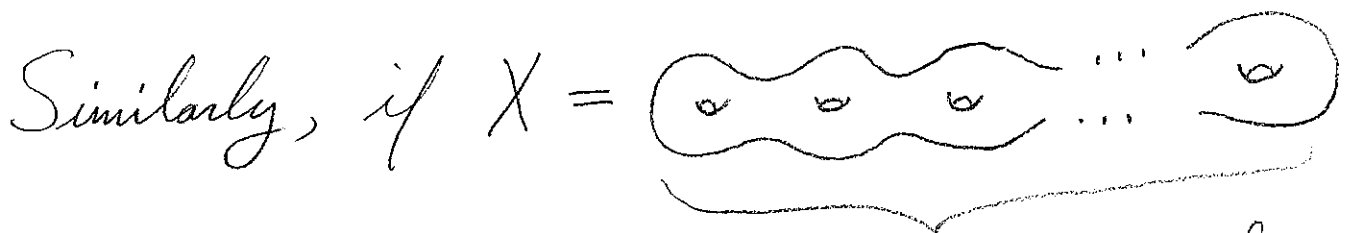
$$\text{deg}(\Delta_{ea}: S^1 \rightarrow X'/bucvd = \text{circle with point } a) = 0$$

Reason:

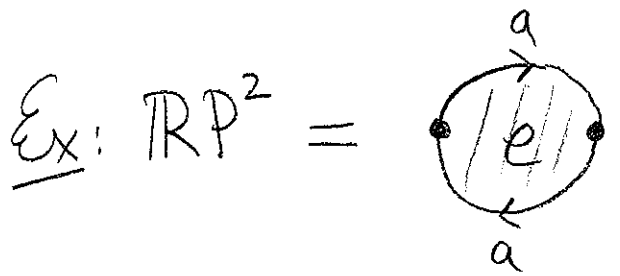


The two points of $\Delta_{ea}^{-1}(y)$ have opposite local degrees, so $\deg \Delta_{ea} = 0$.

So $H_n(X) \cong C_n^{CW}(X) = \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z}^4 & n=1 \\ \mathbb{Z} & n=0 \end{cases}$



$H_n(X) = \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z}^{2g} & n=1 \\ \mathbb{Z} & n=0 \end{cases}$ g holes, comes from a $4g$ -gon

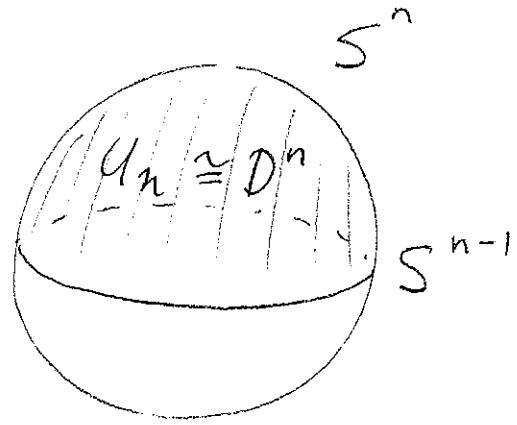


CW chains: $0 \rightarrow \mathbb{Z} \xrightarrow[\times 2]{d_2} \mathbb{Z} \xrightarrow{d_1=0} \mathbb{Z} \rightarrow 0$

$d_2(e) = 2a$ $H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=1 \\ \mathbb{Z} & n=0 \\ 0 & \text{otherwise} \end{cases}$

Ex: $\mathbb{R}P^n = S^n /_{x \mapsto -x}$

Let $g: S^n \rightarrow \mathbb{R}P^n$

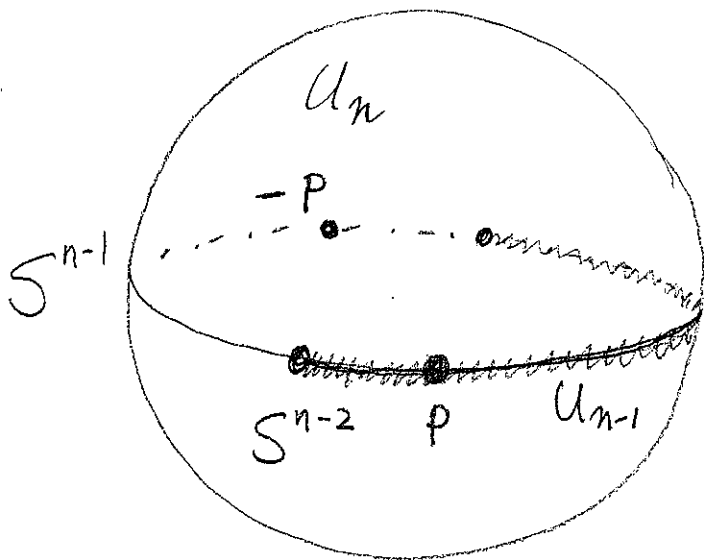


$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup g(U_n)$

and $g|_{\text{int}(U_n)}$ is 1-1. Inductively this gives a CW str to $\mathbb{R}P^n$, with 1 cell in each dim.

Computing the boundary map

$\Delta: \partial U_n \rightarrow X^{n-1} / X^{n-2} = \mathbb{R}P^{n-1} / \mathbb{R}P^{n-2} \cong U_{n-1} / \partial U_{n-1} \cong S^{n-1}$



Pick $\bar{p} \in \text{int}(g(U_{n-1}))$

Then $\Delta^{-1}(\bar{p})$ consists of two points p and $-p$

$\deg_p \Delta = 1$

antipodal map

$\deg_{-p} \Delta = \deg_{-p} \Delta \circ A$

$= (\deg_p \Delta) \deg_{-p} A = (-1)^{n+1}$

So $d_n = 0$ when n is odd
 $d_n = \pm 2$ when n is even

Thus

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } k=n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 & \text{if } k < n \text{ and } k \text{ is odd} \\ \mathbb{Z} & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$$

Since

n odd

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\quad n \quad} \mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z} \xrightarrow{\quad \times 2 \quad} \mathbb{Z} \longrightarrow \dots$$

n even

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\quad \times 2 \quad} \mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z} \xrightarrow{\quad \times 2 \quad} \dots$$