Math 157a Homework #6; Due Monday, February 16

1. Let (M, g) be a Riemannian manifold and p a point in M. Let O be the orthogonal group of $T_p(M)$ (that is, linear automorphisms of $T_p(M)$ which preserve g_p). Give O a biinvariant Riemannian metric so that the associated volume form dV has mass 1, that is $\int_O 1\, dV = 1$; as mentioned in class dV is independent of choice of metric. Give $G = G^2 T_p M$ the measure dm which is the push forward of dV under the quotient map $O \to G$. Show that

$$\int_G K(P)\, dm(P) = \frac{1}{n(n-1)} Scal_p$$

where this integral represents the average sectional curvature at p, and $Scal_p$ is the scalar curvature.

2. Suppose that G is a Lie group and let $\mathfrak{g} = T_e G$ be the Lie algebra. The center of $\mathfrak{g}$ is those elements C such that $[C, X] = 0$ for all $X \in \mathfrak{g}$.

   (a) Suppose that G has a biinvariant metric, is connected, and $\mathfrak{g}$ has trivial center. Prove that G is compact.

   (b) Use part (a) to show that $\mathrm{SL}_2\mathbb{R}$ does not have a biinvariant metric.

3. Let M be an odd dimensional compact Riemannian manifold. Show that if all sectional curvatures are positive, then M is orientable.

4. Let g be a complete Riemannian metric on $\mathbb{R}^2$. For $(x, y) \in \mathbb{R}^2$ let $K(x, y)$ be the sectional curvature at (x, y). Prove that

$$\lim_{r\to\infty} \left(\inf\{K(x, y) \mid x^2 + y^2 \geq r^2\} \right) \leq 0.$$

5. Let (M, g) be a Riemannian manifold which is algebraically locally symmetric (recall from the last HW this means $DR = 0$ everywhere). Let $c\colon [0, \infty) \to M$ be a geodesic, and set $p = c(0)$ and $v = c'(0)$. Define a linear transformation $K_v\colon T_pM \to T_pM$ by $K_v(x) = R(v, x)v$.

   (a) Show that there exists an orthonormal basis e_i of T_pM and $\lambda_i \in \mathbb{R}$ so that $K_v(e_i) = \lambda_i e_i$ for all i.

   (b) Extend the basis e_i of part (a) to parallel vector fields X_i along c. Show that for each t:

$$K_{c'(t)}(X_i(t)) = \lambda_i X_i(t),$$

   where $K_{c'(t)}$ is defined analogously to K_v and the λ_i are the (fixed) eigenvalues of the original K_v.

   (c) Show that the conjugate points of p along c are given by $c(\pi k \lambda_i^{-1/2})$ where k is a positive integer and λ_i is a positive eigenvalue of K_v. (Hint: use (b) to concisely express the Jacobi equation.)