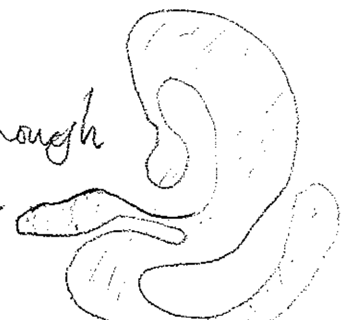
Caltech Winter 2006
Ma 109b

Introduction to Geometry and Topology:

- Welcome! will discuss handout at end
- Note HW #1 on Handout.

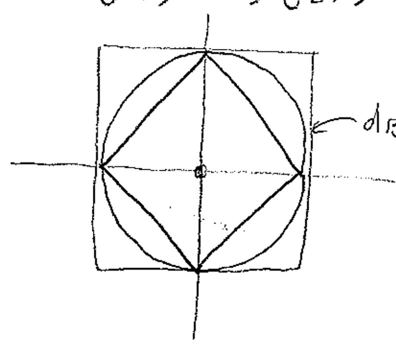


Topology: Study of spaces up to homeomorphism, as though made of rubber. 

Topologist: someone who can't tell a coffee cup from a doughnut.

Geometry: Study of spaces w/ distance functions (metric spaces).
[Where this dist fn is important, not just a source for the topology]

Ex: See above. Ex: $\mathbb{R}^2 = d((x_1,y_1),(x_2,y_2)) =$
- Euclidean $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$
- $|x_1-x_2| + |y_1-y_2|$
- $\max(|x_1-x_2|, |y_1-y_2|)$

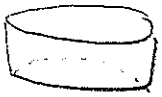

dist 1 from 0

[In both cases, our focus will be on surfaces.]

Def: A surface is a Hausdorff top space X s.t. every pt has an open nbhd homeomorphic to $\mathbb{R}^2$.

Query

Ex: $\mathbb{R}^2$, , , , P^2, K

Non Ex: $\mathbb{R}^3$,  ← almost! a surface w/ ∂.

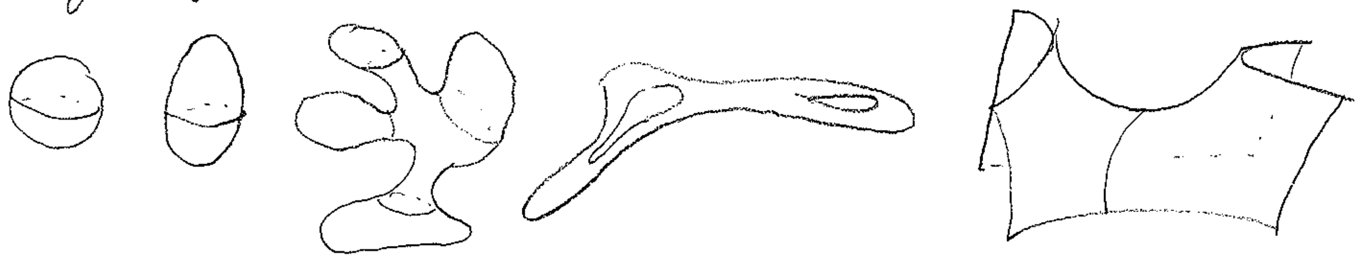
Classification of Surfaces: Any compact surface X is homeomorphic to exactly one of

[Discuss, history, etc. Will start off with this.]

, ...

P^2, $K = P^2 \# P^2$, $P^2 \# P^2 \# P^2$, ...

Geometry of Surfaces: $X^2 \subseteq \mathbb{R}^3$, smoothly embedded

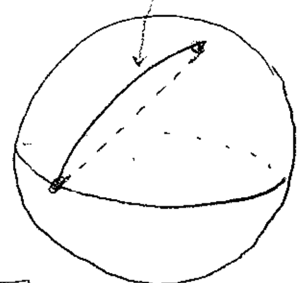


Extrinsic: how X sits in $\mathbb{R}^3$

vs. $\Downarrow$

Intrinsic: distances measured within the surface

shortest path

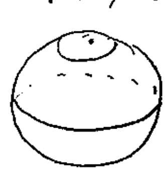


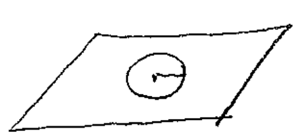
[Different surfaces w/ same intrinsic geometry

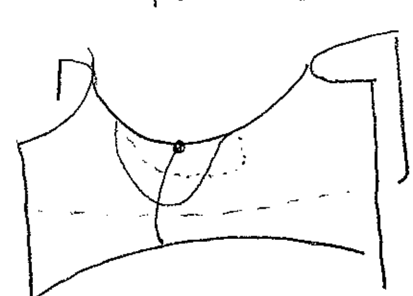
 vs. 

] — consider omitting

[Gaussian] Curvature:

$K > 0$ 

$K = 0$ 

$K < 0$ 

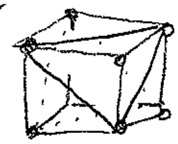
Talk about: 1) actually intrinsic, vs. mean curvature in soap films

2) is a fn of the pt 

Connection between: Topology and geometry:

Euler Char: T a triangulated surface

$$\chi(T) = \#\text{ of verts} - \#\text{ of edges} + \#\text{ of triangles.}$$

$\chi($  $) = 4 - 6 + 4 = 2$ $\chi($  $) = 8 - 18 + 12 = 2!$

Thm: T_1 and T_2 are triang. of the same surface S.

$$\chi(T_1) = \chi(T_2)$$

Def: $\chi(X) = \chi(T)$.

$\chi($  $) = 2$

$\chi($  $) = 0$

$\chi($  $) = -2$

Gauss-Bonnet: Suppose X is a cpt surface S in $\mathbb{R}^3$. Then

$$(\text{Average of } K)(\text{Area of } K) = 2\pi\,\chi(S)$$

Cor: Any  in $\mathbb{R}^3$ has pos curve somewhere

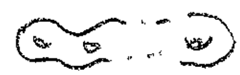
Any  in $\mathbb{R}^3$ has neg curv. somewhere.

Discuss syllabi, policies, etc.

Lecture 2: • Note handout. Contains 1^{st} HW due next Wed.

Last time: Def: A surface is a top space s.t. each pt has a open nbhd $\cong \mathbb{R}^2$.

Classification Thm: A cpt surface is homeo to exactly one of

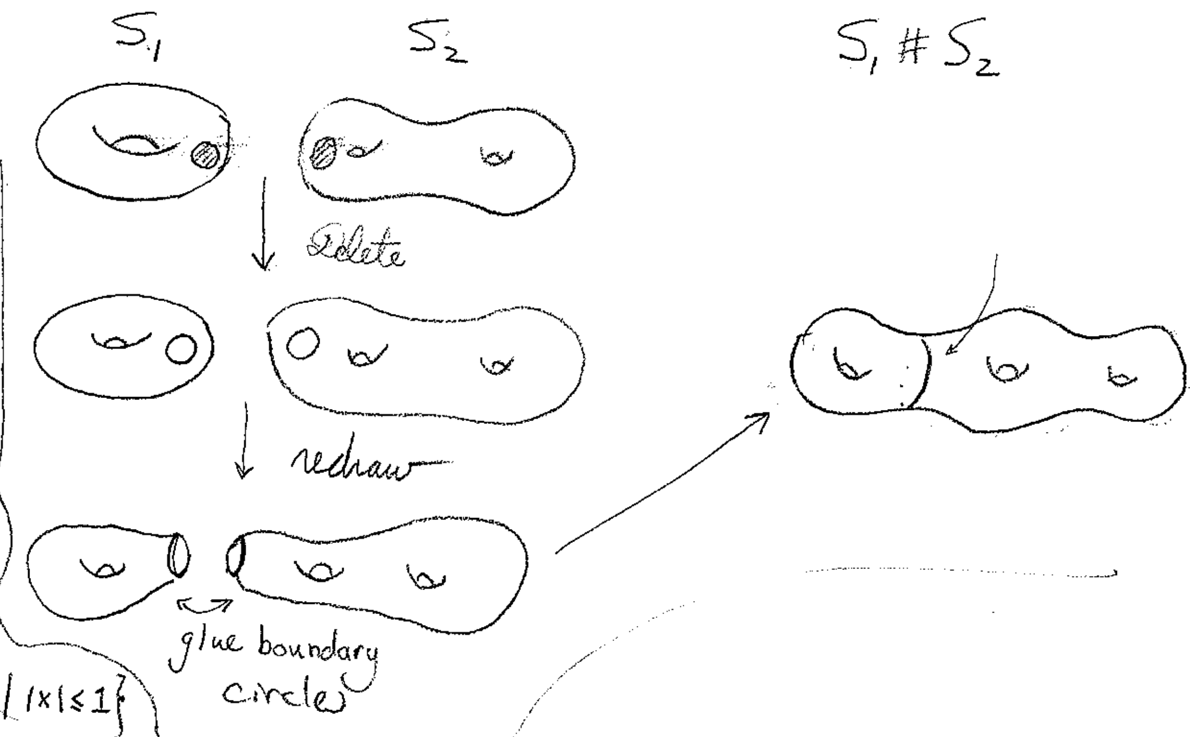
, $T =$ , $T \# T =$ , ..., $T \# \cdots \# T =$ 

P, $K = P \# P$, $P \# P \# P$, ...

Today: 1) Connected sum (#)

2) Jordan curve thm and other top. issues.

Connected sum:



Def: A chart for a surface S is an open set $\cong \mathbb{R}^2$.

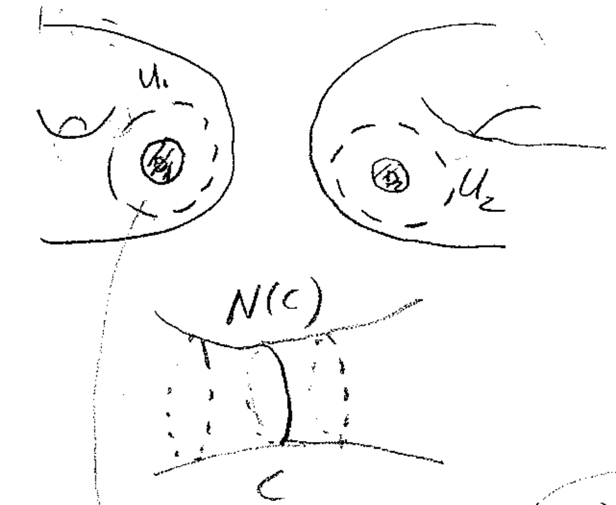
Def: A disc in S is a cpt subset D s.t. $\exists$ a chart $U \supseteq D$ where $U \xrightarrow{\cong} \mathbb{R}^2$ takes $D \longrightarrow \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$

Def: S_1, S_2 surfaces. Then $S_1 \# S_2 = (S_1 \setminus \mathring{D}_1) \cup_f (S_2 \setminus \mathring{D}_2)$ where $D_i \subseteq S_i$ is a disc and $f: \partial D_1 \longrightarrow \partial D_2$ is a homeomorphism.

Need to check: 1) $S_1 \# S_2$ is a surface

2) $S_1 \# S_2$ is indep of the choices of D_i, f.

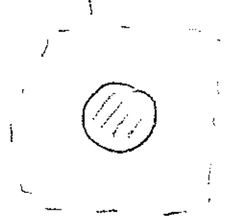
For 1) need to check that pts along the join C have nbhds $\cong \mathbb{R}^2$



$$N(C) = \underbrace{U_1 \setminus \mathring{D}_1}_{\cong S^1\times[0,\infty)} \cup_f \underbrace{U_2 \setminus \mathring{D}_2}_{\cong S^1\times[0,\infty)}$$

gluing $S^1\times\{0\}$ to $S^1\times\{1\}$ by some homeo.

$$\cong S^1\times(-\infty,\infty).$$



2) breaks into 2 issues

a) coarse: fund diff choices for f — id, reflection

$\mathbb{Z} = \pi_1(S^1) \longrightarrow \pi_1(S^1) = \mathbb{Z}$

$id_*(1) = 1$

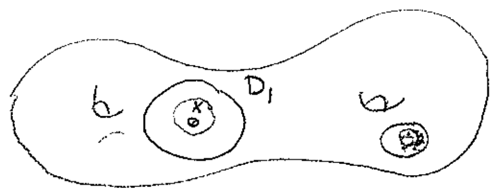
$r_*(1) = -1$

b) subtle: choices of D_i

[a) is in some sense "accidental", a consequence of the dimension]

[I will deal with 2) in an odd way — I will prove the classification thus avoiding this issue.]

b) <u>Thm</u> $D_1, D_2 \subseteq S$ then $\exists\, S \xrightarrow[\cong]{f} S$ s.t. $f(D_1) = f(D_2)$



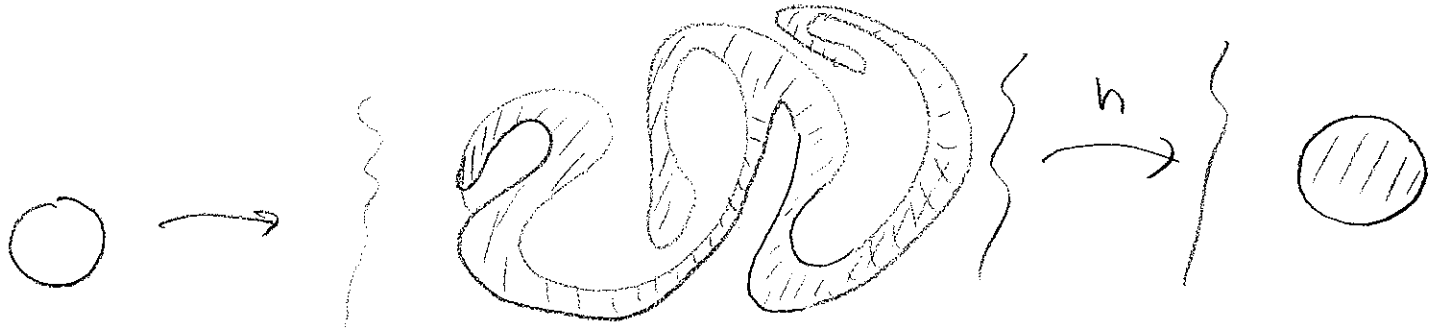
<u>Lemma1</u>: True if $D_2 \subseteq \mathring{D}_1$

<u>Lemma2</u>: given $x, y \in S$, $\exists$ $S \xrightarrow[f]{\cong} S$ s.t. $f(x) = f(y)$

Lemma 1 is on HW, but real work comes from:

Schönflies Thm: Let C be the image of $f: S^1 \xhookrightarrow{1-1} \mathbb{R}^2$.

Then $h: \mathbb{R}^2 \circlearrowleft \cong$ s.t. $h(C)$ is the unit circle $\{x \mid |x| = 1\}$.



Jordan Curve Thm: Any circle C as above separates $\mathbb{R}^2$ into 2 regions. [Section 5.6 of Armstrong.]

Remarks: [I was pretty non-impressed by these thms.]

1) Continuous functions are really messy. [see handout on next page]

2) Analog is not true in dim 3! $\exists\ f: S^2 \hookrightarrow \mathbb{R}^3$
   s.t. some comp of $\mathbb{R}^3 \setminus f(S^2)$ is not simply connected!
   See other side of handout, discuss

Thm: Any cpt surface S has a triangulation.] if time

$\boxed{\text{Comment on differing w/ text.}}$

Lecture 3: Today and Wed: proving

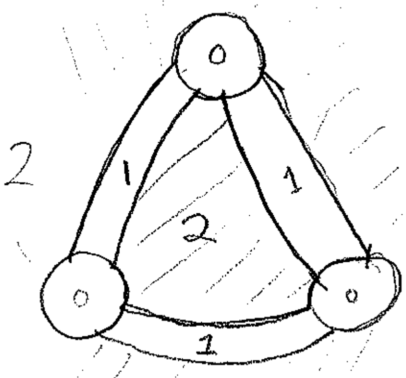
Thm: Any cpt, connected surface is homeo to exactly one of

1)  , ...

2) P, $P \# P$, $P \# P \# P, \ldots$

[Will use diff proof than either text, say why.]

Handle Decomposition: 0) Start w/ 0-handles $= D^2$

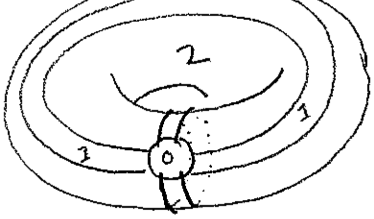


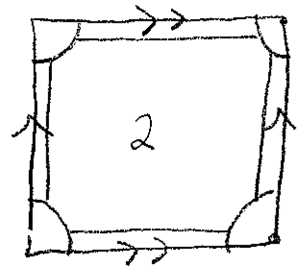
1) Glue 1-handles $= I \times I$ to the boundry of the 0-h along $\partial I \times I$

2) Glue 2-handles $= D^2$ along whole of ∂D^2  ...to every boundry comp of 0-1 handles

$\Rightarrow$ gives a surface.

Ex:

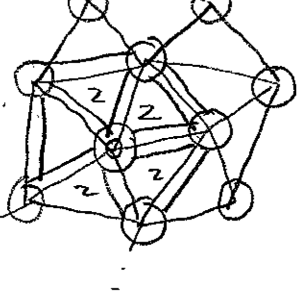




Lemma: S cpt conn surf. Then S has a handledecomp with only one 0-handle and 2-handle. [and some unknown # of 1 handles]

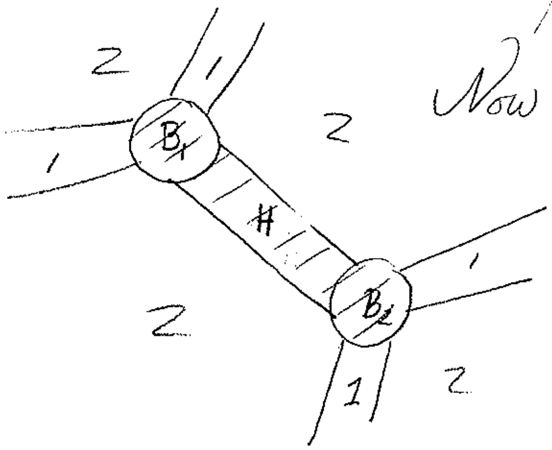
Pf: As S has a triangulation, it has a handle decomp.

Draw large



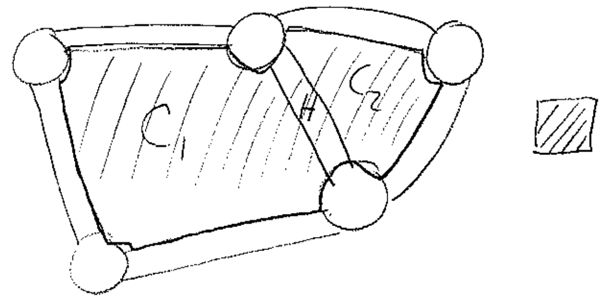
Suppose there are at least two 0-handles. Then $\exists$ two distinct 0-handles B_1, B_2 joined by a 1-handle

[ask class why. Reason: otherwise disconnected as adding 2-handle doesn't change the # of conn. comp.]

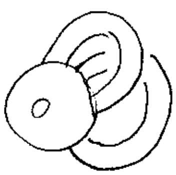


Now consider $B_1 \cup B_2 \cup H$ as a 1-handle. [Check still have a handle decomp]

Similary if have multiple 2-handles, find two adjacent across a 1-handle and then ammalgamate

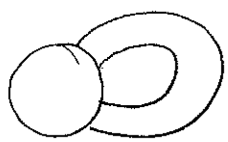


To record such a hand. decomp. just need to draw 0 and 1 cells.

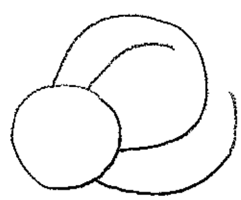


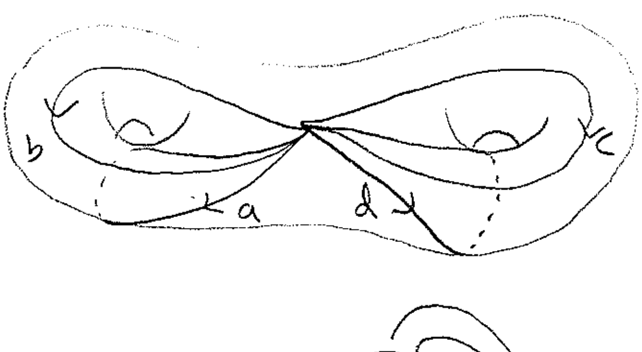
+ one 2-handle $\overset{[\text{Query}]}{=}$ T^2

Two kinds of 1-handle



vs.





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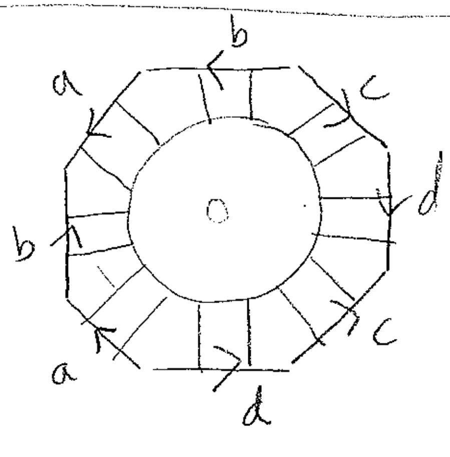

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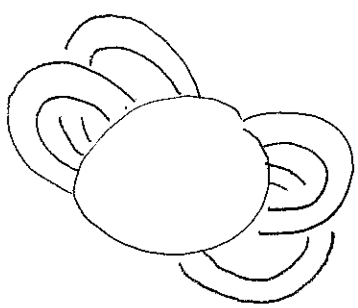
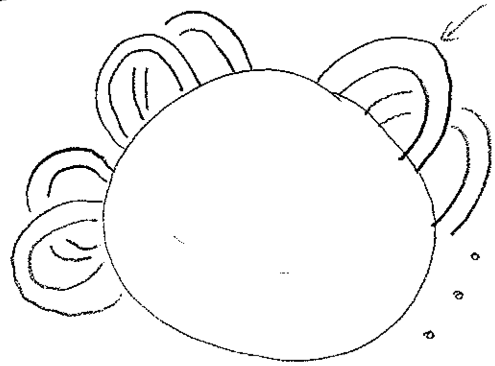


+ 2 handle.

Def: [To avoid # sum issue]

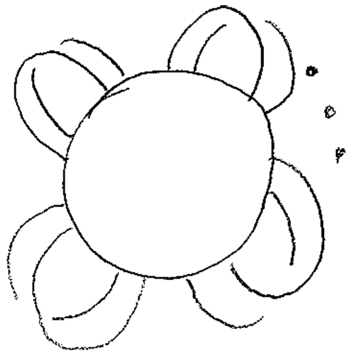
$\underbrace{T \# \cdots \# T}_{n \text{ times}} :=$



n of these.

+ 2 handle

$\underbrace{P \# \cdots \# P}_{n} =$



+ 2-handle.

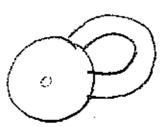
Pf of Class Thm: Consider a handle decomp of S w/ one 0 handle, one 2-handle and n 1-handles.

Orientable case: no band is twisted.

Claim: $S \cong \underbrace{T \# \cdots \# T}_{n/2}$

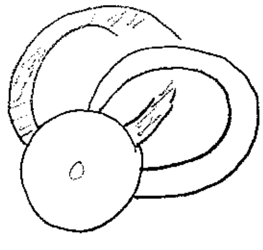
$n = 0$: $0 + 2 = S^2$

$n = 1$:



← not allowed as we would have to add two 2-handles to make a surface.

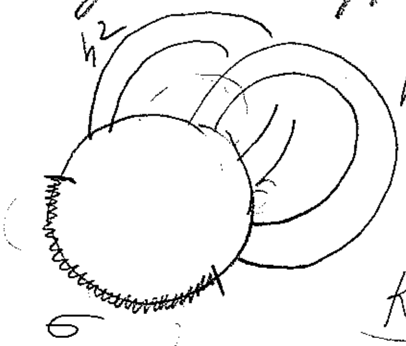
$n = 2$



So we don't have the same problem as before, 2nd 1-handle must be

$= T$ ✓

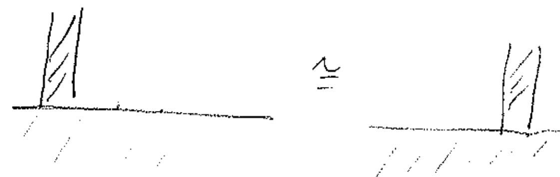
In general, suppose we have 1-handles $h^1, h^2, \dots, h^n$

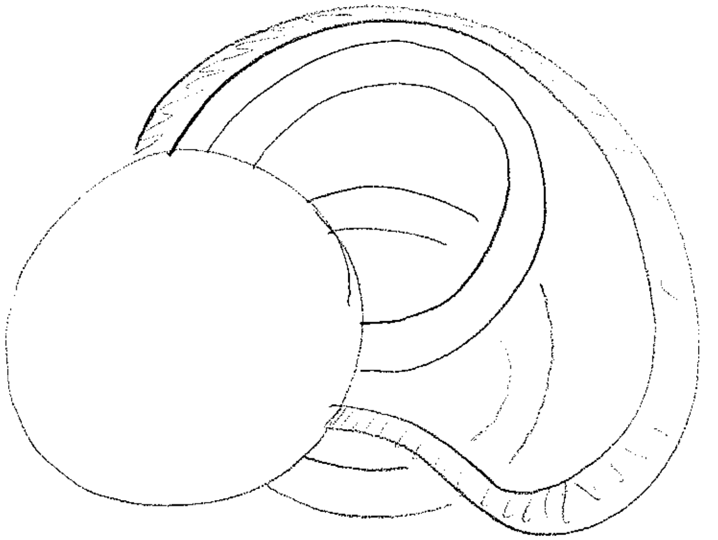


There must be some h^i, say h^2, going from one bdry comp to the other

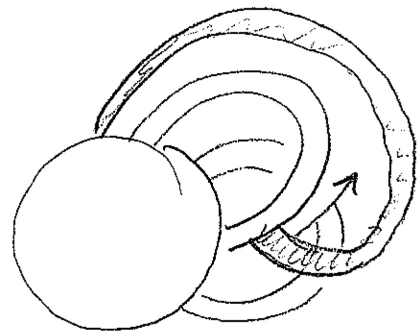
Key claim: Can assume h^i for $i > 2$ are all attached out here

Pf: use handleslide

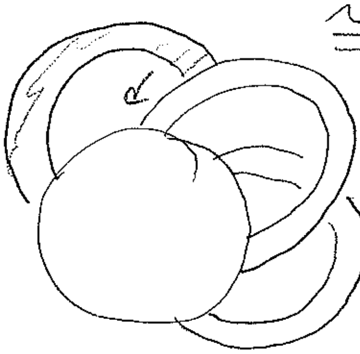




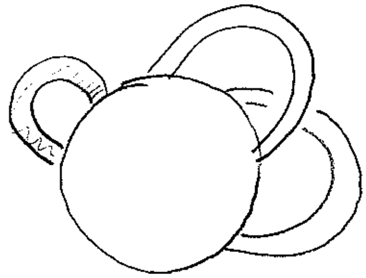
$\cong$



$\cong$



$\cong$



Point: the boundary of

is just a circle

Now repeat the argument on $h^3, \dots, h^n$ taking care to never leave them outside the region σ.

This concludes the orientable case. [Query: Abel. π_1]

<u>Lecture 4</u>: On side board: Class. thm, lemma on existence of handle decomp, def of $T\#\cdots\#T$, $P\#\cdots\#P$ as handles + discs w/ ident.

<u>Pf of Class</u>: S a cpt conn surface. Choose a handle decomp w/ one 0-handle and one 2-handle.

<u>Case 1</u>: There are no twisted 1-handles.

<u>Claim</u>: If there are n 1-handles $h_1, \dots, h_n$ then $S \cong \underbrace{T\#\cdots\#T}_{n/2}$

$X_0 \quad X_1 = X_0 \cup h_1 \quad X_2 = X_1 \cup h_2 \quad \dots \quad X_n \qquad S = X_n \cup D^2$

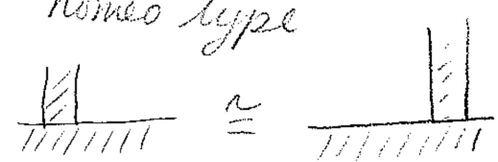
1) Doesn't matter what order we add the handles.

2) Can change handle structure of X_K using handleslides

$N(\partial X_K) \cong \partial X_K \times [0,1]$

[Query: Not a handle decomposition.]

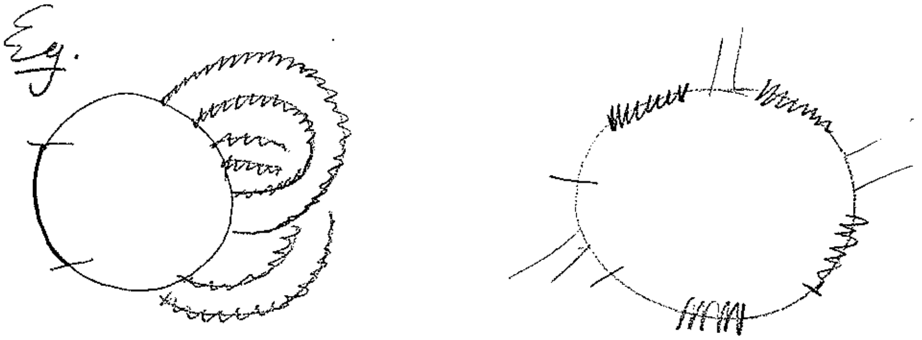
Doesn't change homeo type



[<u>Note</u>: May have to move $h_{K+1}, \dots, h_n$ slightly to get a handle decomp at the next stage.]

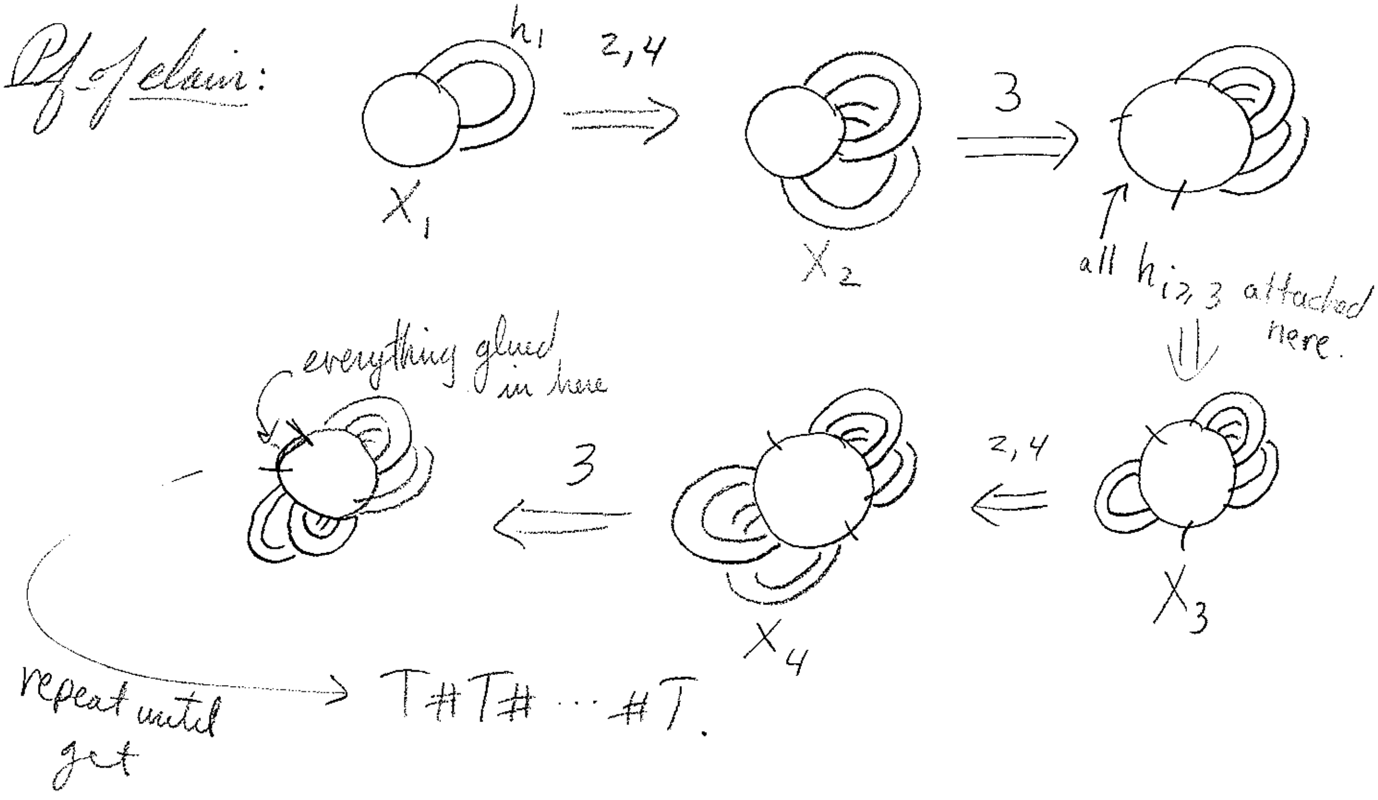
3) If ∂X_K is connected — consists of just one circle — then we can assume all remaining handles are glued to a segment of ∂X_K we get to choose

Eg.



4) If ∂X_K has two components C_1 and C_2 then at least one $h_{K+1}, \dots, h_n$ has one end on C_1 and the other on C_2.

Pf of claim:



Claim: Suppose there are n 1-handles at least one of which is twisted. Then $S \cong \underbrace{P \# \cdots \# P}_{n}$.

Pf: HW.

To complete the proof need to show all these are distinct.

$$\pi_1(\#_n T) = \langle a_1, b_1, \dots, a_n b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1 \rangle.$$

$$\pi_1(\#_n P) = \langle a_1, \dots, a_n \mid a_1^2 a_2^2 \cdots a_n^2 = 1 \rangle.$$

<u>Problem</u>: These a different, but how do we prove it?

<u>Def</u>: G is a gp. Then $G' =$ gp gen by $\overset{g_1 g_2 g_1^{-1} g_2^{-1}}{\overset{\shortparallel}{[g_1, g_2]}}$ for all $g_1, g_2 \in G$

Set $G^{ab} = G/G'$, the abelianization of G. [why is this a normal subgp? why abelian?]

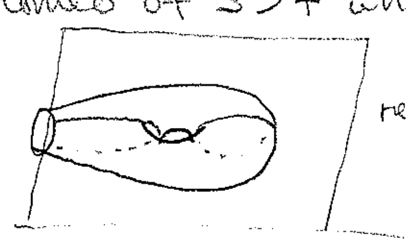
$$\pi_1(\# T^n) = \mathbb{Z}^{2n} \qquad \pi_1(\#_n P) = \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2.$$

To compute, add relations so that all the gens commute.

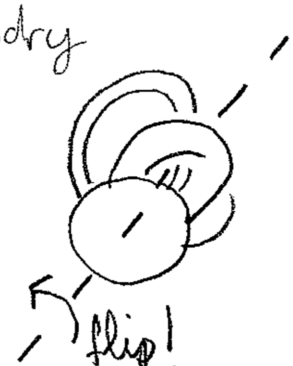
Why is # sum well-defined?

course issue: two diff homeos of $S^1 <\begin{smallmatrix} id \\ r = \text{reflection} \end{smallmatrix}$

<u>Point</u>: S a compact connected surface w/ one boundry circle, then $\exists$ a homeo of $S \supset f$ which flips the ∂ circle:

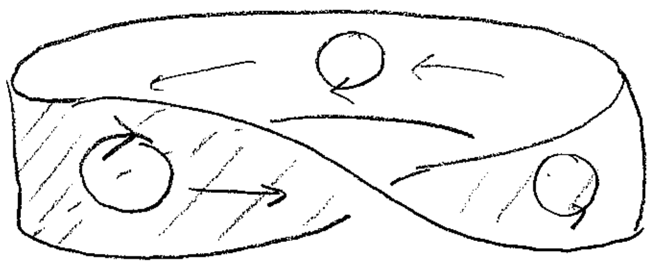


reflect. or

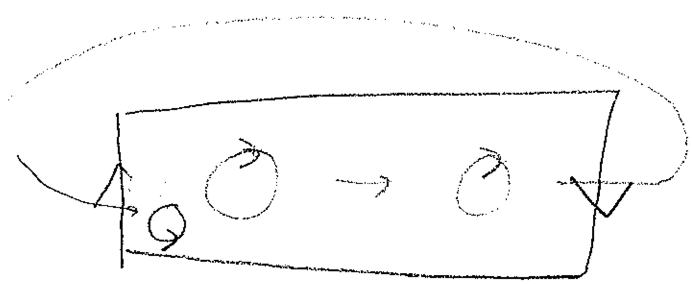


works for any $T \# \cdots \# T$

If surface contains a Möbius band, e.g it has a twisted handle, then just slide the boundary circle around the band



to reverse the orientation.



—————— ∘ ——————

Lecture 5: Smooth surfaces in $\mathbb{R}^3$

Def: If $U \subseteq \mathbb{R}^n$ then $f : U \to \mathbb{R}^m$ is smooth if

- U is open [need to sense of $\frac{\partial}{\partial x}$]
- all partial derivatives of f of all orders exist and are continuous.

The derivative of f at p is
$$D_p f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{pmatrix}.$$
where $f(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$

Gives best linear approximation to f

$$f(x) = f(x_0) + (D_{x_0} f)(x - x_0) + E(x - x_0)$$

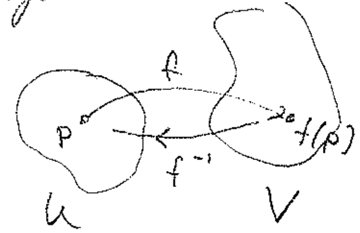
where $\exists \delta, M$ s.t. $|E(x-x_0)| \leq M |x - x_0|^2$ for all $|x - x_0| < \delta$.

Def: U, V open sets in $\mathbb{R}^n$. A fn $f : U \to V$ is a diffeomorphism if it is bijective and f, f^{-1} are both smooth. [invertible, full rank]

[A kind of homeomorphism]

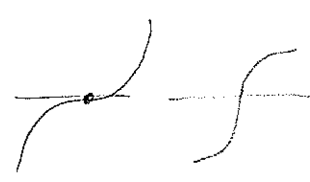
Note: If f is a diffeo, then $\forall p \in U$, $D_p f$ is non-singular.

Pf: $D_{f(p)} f^{-1} \circ D_p f = D_p (f^{-1} \circ f) = D_p(\mathrm{Id}) = I$.



Ex: A non-diffeo: $f : \mathbb{R} \circlearrowleft$ $\quad f(x) = x^3$ a smooth homeo

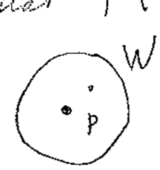
$f^{-1}(x) = x^{1/3}$ not diff at 0.



Inverse Function Thm: $f : (U \subseteq \mathbb{R}^n) \to \mathbb{R}$ smooth.

If $p \in U$ is such that $D_p f$ is invertible

$\exists$ a open nbhd W of U such that $f(W)$ is open and $f : W \to f(W)$ is a diffeomorphism.



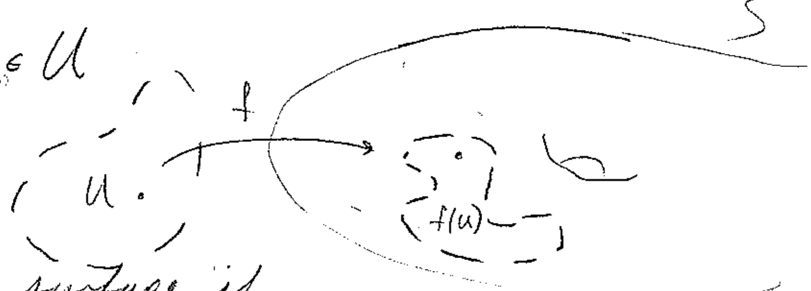
Def: $U^{open} \subseteq \mathbb{R}^2$, $f : U \to S \subseteq \mathbb{R}^3$ a smooth map.

Then f is a coordinate patch if

1) f is a homeo from U to $f(U)$, and $f(U)$ is open in S.

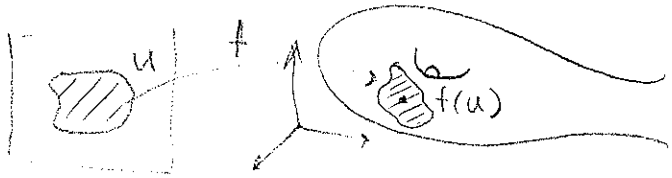
2) $D_p f$ is 1-1 for each $p \in U$

[diffeo-like, let us define a tangent plane.]



Def: $S \subseteq \mathbb{R}^3$ is a smooth surface if for each $p \in S$, there is a coordinate patch $f : U \to S$ with $p \in f(U)$

Note: such an S is also a topological surface. in the old sense [note that $U \not\cong \mathbb{R}^2$]



[Def differs from text, doesn't require it is a homeo.]

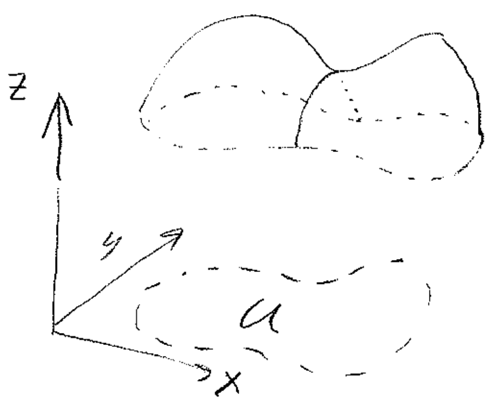
Ex: $U \overset{open}{\subseteq} \mathbb{R}^2$, $h : U \to \mathbb{R}$ smooth fn

Monge Patch

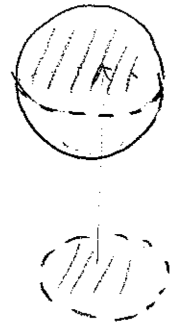
$$S = \{(x, y, h(x,y)) \mid (x,y) \in U\}$$

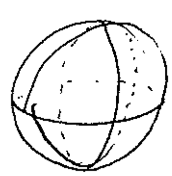
$f(x,y) = (x, y, h(x,y))$ a coor. patch.

$$Df = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix}$$



Ex:





$h = \sqrt{1 - (x^2 + y^2)}$

<u>Lecture 6</u>: <u>Last time</u>: Def of smooth surface

<u>Today</u>: · Change of coor. lemma

· smooth maps between surfaces, diffeo

<u>Change of Coor Lemma</u>: $S \subseteq \mathbb{R}^3$ a smooth surface.

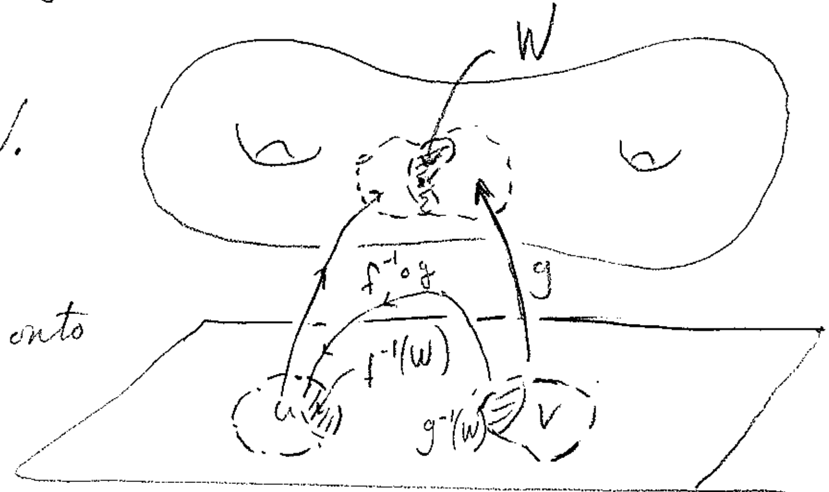
write up ahead of time

Let $f: U \to S$, $g: V \to S$ coordinate charts

Set $W = f(U) \cap f(V)$. Then $f^{-1} \circ g : g^{-1}(W) \to f^{-1}(W)$ is smooth

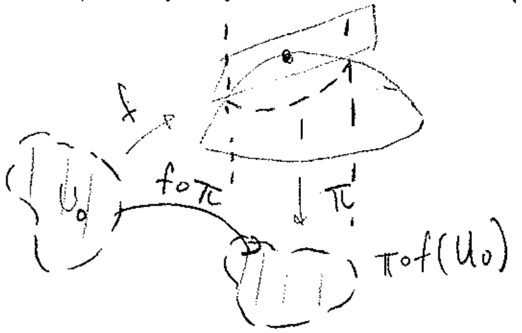
<u>Pf</u>: First, suppose f is a Monge chart, as in the HW.

Say $f(x,y) = (x, y, h(x,y))$, and let $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ be proj onto the xy plane. Then



$f^{-1} \circ g = \pi \circ g$ which is the composit of two smooth functions, hence smooth. In general, we need

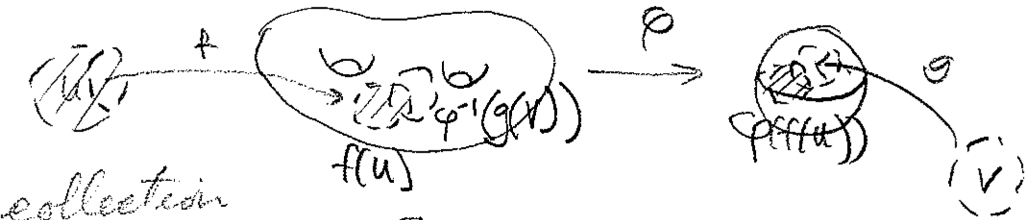
<u>Sublemma</u>: f a coor patch, p a point in $f(U)$. Then $\exists\, U_0 \overset{\text{open}}{\subseteq} U$ w/ $f(U_0) \ni p$ and a proj fn π s.t. $\pi \circ f$ is a <u>diffeo</u> on U_0.



Thus $(\pi \circ f)^{-1} \circ \pi$: open in $\mathbb{R}^3 \to U_0$ is a smooth fn and restricted to $f(U_0)$ it is just f^{-1}! This case is then just the same as before. ∎

Def: S_1, S_2 are smooth surfaces. Then $\varphi: S_1 \to S_2$ is smooth if $\forall$ coor patches $f: U \to S_1$, $g: V \to S_2$ we have $g^{-1} \circ \varphi \circ f : (\varphi \circ f)^{-1}(g(V)) \longrightarrow V$ is smooth.

[By change of coor lemma, only need to check cond for some collection of charts which cover S_1 and S_2]



Def: A map $\varphi: S_1 \to S_2$ is a diffeomorphism if it is a bijection and φ and φ^{-1} are smooth.

[smooth analog of homeo]:

Q: Is every topological surface a smooth surface in $\mathbb{R}^3$?

A: [Query] No. After all some don't even embed topologically in $\mathbb{R}^3$, e.g. P or K. [But this is a silly reason could just use $\mathbb{R}^4$]

Abstract Smooth Surface: is a topological surface S with a collection of homeos $f_\alpha : (U_\alpha \overset{\text{open}}{\subseteq} \mathbb{R}^2) \longrightarrow$ (open subset of S) s.t. $f_\beta^{-1} \circ f_\alpha : f_\alpha^{-1}(f_\beta(U_\beta)) \longrightarrow f_\beta^{-1}(f_\alpha(U_\alpha))$ is smooth.

[I.e. defining exactly so the change of coor lemma holds]

Eg.

<u>Thm</u>: S a topological surface. Then there exist a coll (f_α, U_α) making it into a smooth surface. Any two such smoothings are diffeomorphic. [Thus class of surfaces doesn't change.]

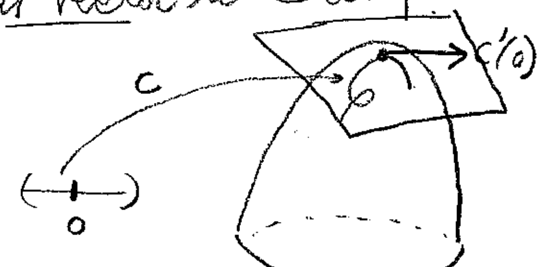
[<u>Pf</u>: For existence, use a triangulation and do the gluing in a controlled way.]

<u>Note</u>: False in higher dimensions: $S^7 = \{x \in \mathbb{R}^8 \mid |x| = 1\}$ [Query] has 28 non diffeomorphic smoothings! Discuss / poss meanings

— o —

Suppose $c : (-\varepsilon, \varepsilon) \to S \subseteq \mathbb{R}^3$ is a smooth curve w/ $c(0) = p$. Then $c'(0) \in \mathbb{R}^3$ is called a <u>tangent vector</u> to S at p.

The collection of all such <u>tangent</u> vectors is the <u>tangent space</u> $T_p S$.

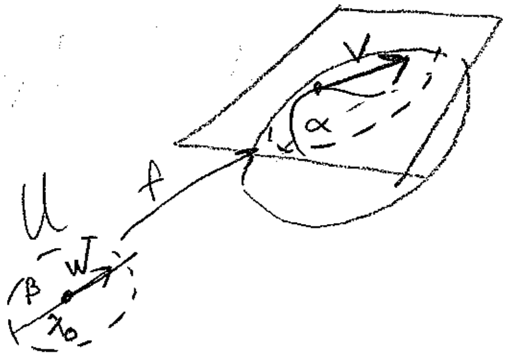


<u>Lemma</u>: Suppose $f : U \to S$ is a coor patch w/ $f(x_0) = 0$

Then $T_p M = \text{image}(D_{x_0} f)$ ← always 2 dim'l.

[In particular, $T_p M$ is a 2 dimensional linear subspace of $\mathbb{R}^3$]

Pf: ($\subseteq$) Suppose $V = D_{x_0}f(W)$. Let $\beta(t) = x_0 + tW$

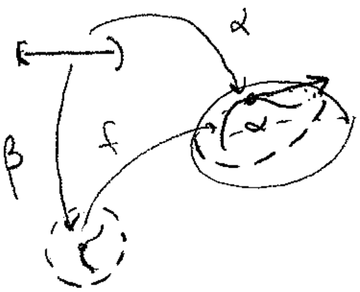


Let $\alpha = f \circ \beta : (-\varepsilon, \varepsilon) \to S$.

Then $\alpha'(0) = D_{x_0}f(\beta'(0))$
$= V$

by chain rule.

($\supseteq$) Let α be a curve defining a tangent vector V, shrink ε so that image lies in $f(U)$.



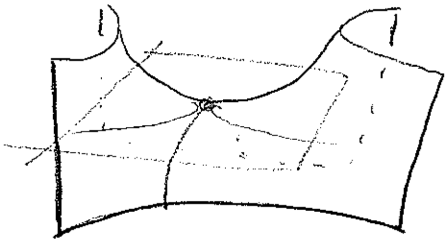
Then $\beta = f^{-1} \circ \alpha$ is smooth and

$\alpha = f \circ \beta$ so

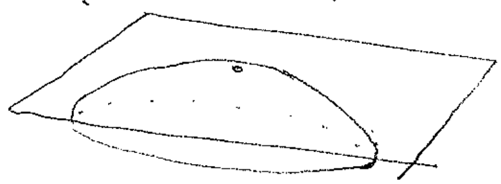
$V = \alpha'(0) = D_{x_0}f(\beta'(0))$ as desired. ∎

Lecture 7: Today: Geometry of curves.

[Quick comment:]



Tangent space = plane that best approximates S at p.



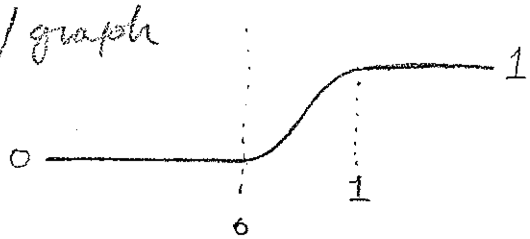
<u>Smooth curve:</u> smooth fn $c:(a,b) \longrightarrow \mathbb{R}^3$.

<u>regular curve:</u> $c'(t) \neq 0$ for all $t \in (a,b)$.

[regular curve is analogous to our smooth surface; only need one chart as the topology of 1 mflds is trivial. Need to avoid.

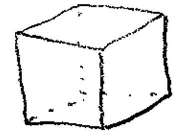
<u>Oddly smooth curve:</u> $c:(-1,1) \to \mathbb{R}^2$ whose image is 

$h: \mathbb{R} \to \mathbb{R}$ smooth w/ graph



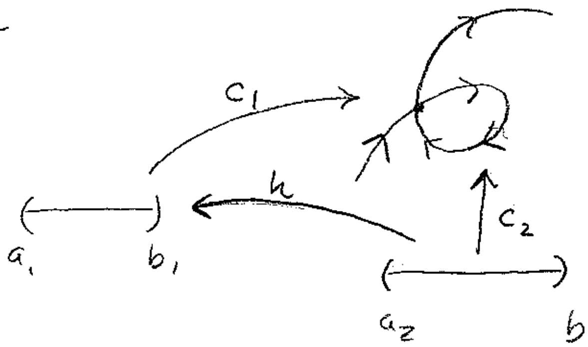
$$h(x) = \begin{cases} 0 & x < 0 \\ \dfrac{f(x)}{f(x)+f(1-x)} & x \in [0,1] \\ 1 & x > 1 \end{cases} \qquad f(x) = e^{-1/x}$$

Take $c(t) = (h(t), h(t+1))$ to trace out.

<u>N.B.</u>: $\exists$ a smooth map  $\longrightarrow \mathbb{R}^3$ w/ image 

Def: A smooth curve $C_2 : (a_2, b_2) \longrightarrow \mathbb{R}^3$ is a reparameterization of $C_1 : (a_1, b_1) \longrightarrow \mathbb{R}^3$ if $\exists$ a diff $h : (a_2, b_2) \longrightarrow (a_1, b_1)$ s.t. $C_2 = C_1 \circ h$

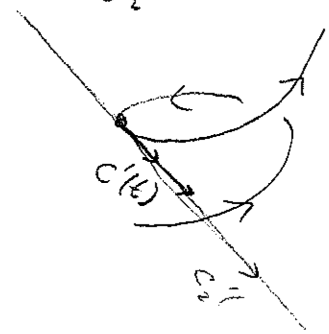
[regard curves as the same if they are reparam.]



[Note: curve not required to be embedded.]

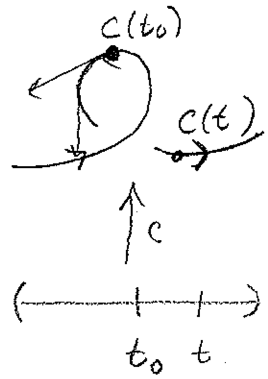
Fix $c : (a,b) \longrightarrow \mathbb{R}^3$ a regular curve

$$T(t) = \frac{c'(t)}{|c'(t)|} \quad \text{unit tangent vector.}$$



[Really what is well defined is the (oriented) tangent line]

Note: Can reparameterize so that c moves at unit speed and $c'(t) = T(t)$.



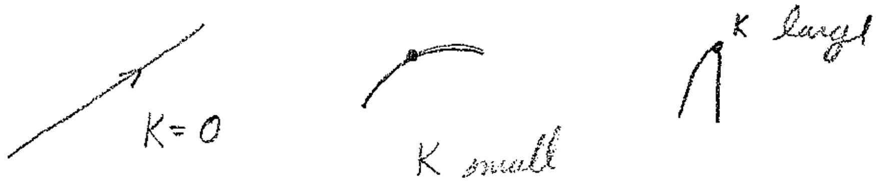
dist (t_0, t_1) along c is $\displaystyle\int_{t_0}^{t_1} |c'(t)|\, dt = d(t)$

d is smooth, strictly increasing, so gives a diffeo $d : (a,b) \longrightarrow (a', b')$

Then $c_u = c \circ d^{-1}$ is a unit speed param.

For now: let's focus on a unit speed curve.

<u>Curvature</u>: qualitative measure of how bent the curve is at each point.

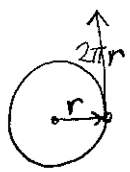


<u>Def</u>: c a unit-speed curve, then $K(t) = |T'(t)|$

$$= |c''(t)|$$

<u>Ex</u>: line has $K=0$

<u>Ex</u>:

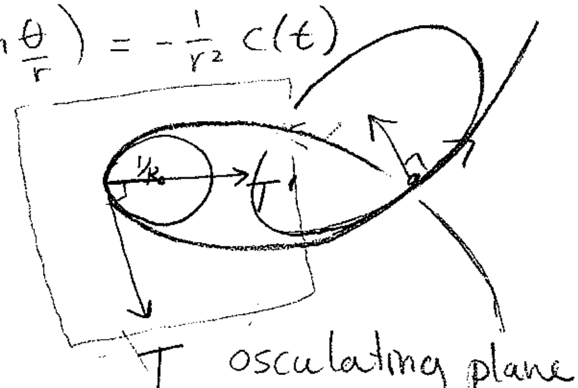


$$c(t) = \left(r \cos\frac{\theta}{r}, \ r \sin\frac{\theta}{r}\right)$$

$$c'(t) = \left(-\sin\frac{\theta}{r}, \ \cos\frac{\theta}{r}\right) \rightsquigarrow \text{unit speed.}$$

$$c''(t) = \left(-\frac{1}{r}\cos\frac{\theta}{r}, \ -\frac{1}{r}\sin\frac{\theta}{r}\right) = -\frac{1}{r^2}\, c(t)$$

$$K = \frac{1}{r}$$



<u>Geometrically</u>: 1) $\frac{1}{K}$ = turning radius

2) $K(t) = \frac{1}{r}$ where r is the radius of the unique round circle at $c(t)$ whose first two derivatives match w/ c', c''.

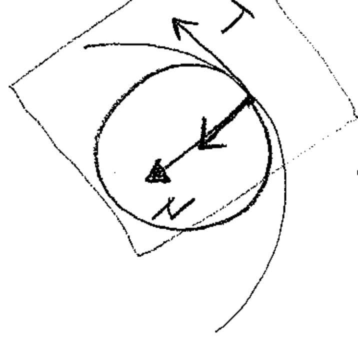
3) K measures change in length as we push in the direction of the normal

} loosy picture.

Detail: [Accelleration is perp to T as speed is not changing]

$$0 = \frac{d}{dt}\langle T,T\rangle = \langle T',T\rangle + \langle T,T'\rangle = 2\langle T,T'\rangle$$

Set $N(t) = \dfrac{T'}{|T'|}$

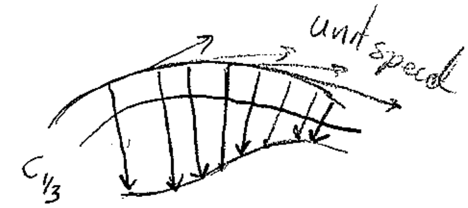


osculating plane

$$\boxed{C'' = K N}$$

Consider the family of curves $c_s(t) = c(t) + s\,N(t)$

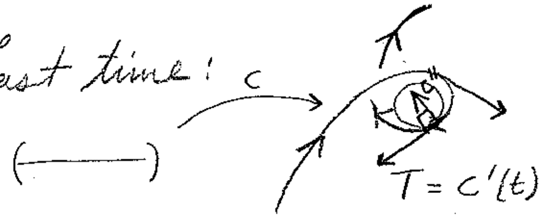
$C : \underset{t}{(a,b)} \times \underset{s}{\mathbb{R}} \longrightarrow \mathbb{R}^3$ a smooth fn.



$$L(s) = \int_a^b \left|\frac{\partial c_s(t)}{\partial t}\right| dt \quad \text{length of } c_s$$

$$\left.\frac{dL}{ds}\right|_{t=0} = -\int_a^b K\,dt$$

Lecture 8: Last time: c



unit speed

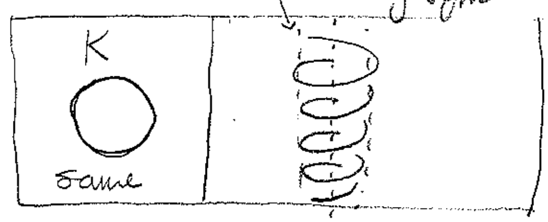
$K(t) = |T'(t)| = |c''(t)|$

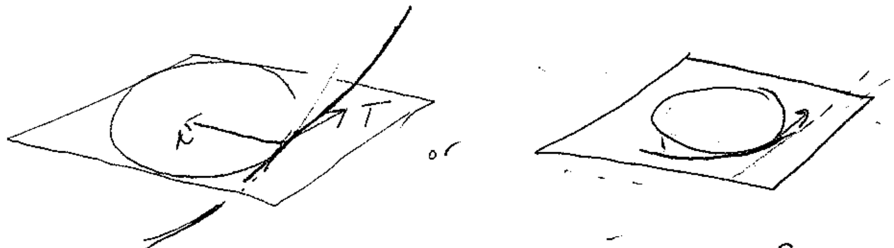
$$N = \frac{c''(t)}{|c''(t)|}$$

$$\boxed{KN = T'}$$

[Curvature measures...]

K captures only part of the info about c.

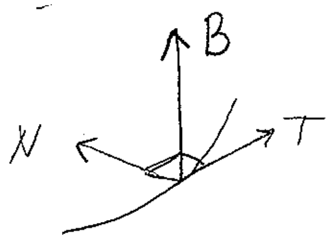




} unmeasured: how c twists w.r.t the osculating plane.

Set $B(t) = T(t) \times N(t)$

[Binormal]



Consider $B'(t)$, and note: $\langle B', B\rangle = 0$ [same reason as last time]

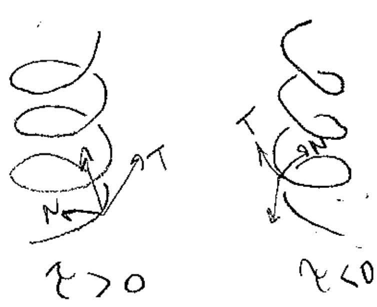
Also $\langle B', T\rangle = 0$ because

$$0 = \frac{d}{dt}\langle B, T\rangle = \langle B', T\rangle + \langle B, T'\rangle = \langle B', T\rangle + \overset{0}{\overset{\shortparallel}{\langle B, KN\rangle}}$$

Hence B' is a scalar mult of N, say

$$B'(t) = -\tau(t)\, N(t)$$

$\longrightarrow$ torsion of c at t.



Note: $\tau(t)$

c is strongly regular if $\kappa(t) \neq 0$ for all t.

Thm: For a strongly regular unit speed curve,

$$(*)\quad \begin{aligned} T' &= \ \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned} \qquad \text{for each } t.$$

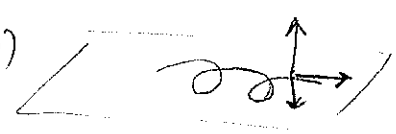
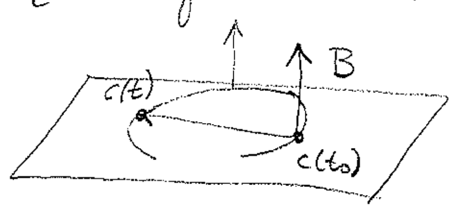
Pf: $\langle N', N \rangle = 0$ for the usual reason.

$$0 = \frac{d}{dt}\langle N, T \rangle = \langle N', T \rangle + \langle N, T' \rangle \Rightarrow \langle N', T \rangle = -\kappa.$$

$$0 = \frac{d}{dt}\langle N, B \rangle = \langle N', B \rangle + \langle N, B' \rangle \Rightarrow \langle N', B \rangle = -\tau. \qquad \blacksquare$$

Q: To what extent does κ, τ determine c?

Ex: c lies in a plane $\Longleftrightarrow \tau = 0$ for all t. $\Longleftrightarrow$ B is const

($\Rightarrow$)  ($\Leftarrow$) 

$$\frac{d}{dt}\langle c(t) - c(t_0), B(t) \rangle = \langle c'(t), B(t) \rangle + \langle c(t) - c(t_0), B'(t) \rangle = 0.$$

$\Rightarrow \langle c(t) - c(t_0), B(t) \rangle = 0 \;\; \forall t$, so lies in a plane. $\blacksquare$

Fundamental Theorem of Curves:

Let $K, \tau : (a,b) \longrightarrow \mathbb{R}$ be smooth fns, w/ $K > 0$.

Then $\exists$ a strongly regular curve $c : (a,b) \longrightarrow \mathbb{R}^3$

w/ curvature and torsion fns equal to K and τ.

This curve is unique up to translation and rotation.

Idea: $(*)$ are a set of differential equations for T, N, B. For general reasons, they have a solution, say w/ init cond

$T(t_0) = (1,0,0)$
$B(t_0) = (0,1,0)$
$N(t_0) = (0,0,1)$

Then set $c = \int_{t_0}^{t} T(t)\,dt$

Check that $T_c, N_c, B_c = T, N, B$.

and T, N, B are orthonormal.

What about non unit speed curves?

$$T = \frac{c'}{|c'|} \qquad B = \frac{c' \times c''}{|c' \times c''|} \qquad N = B \times T$$

$$K = \frac{|c' \times c''|}{|c'|^3} \qquad \tau = \frac{\langle c' \times c'', c''' \rangle}{|c' \times c''|^2}$$

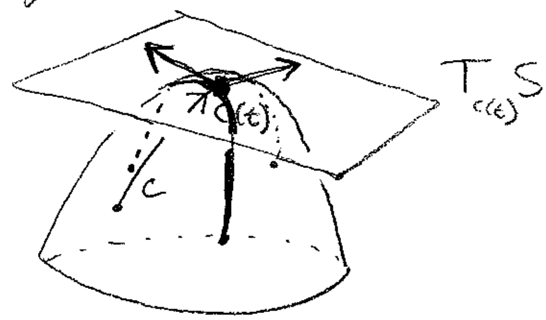
The reason that you don't solve for N in terms of c'' is that if $c = \underbrace{c_u \circ g}_{\text{unit speed}}$

$c'' = \left(c_u'(g(t))\, g(t) \right)' = c_u''(g(t))(g'(t))^2 + c_u'(g(t))\, \overset{\text{don't know}}{g''(t)}$

Lecture 9: Today: Length and area of surfaces.

Length:

$c : (a,b) \longrightarrow S$ curve in surface S.



length of $c = \int_a^b |c'(t)|\, dt \qquad c'(t) \in T_{c(t)}S$

[Can also talk about angles of vectors in $T_{c(t)}S$; both are encoded in this inner product]

[intrinsic geom all comes from this information.]

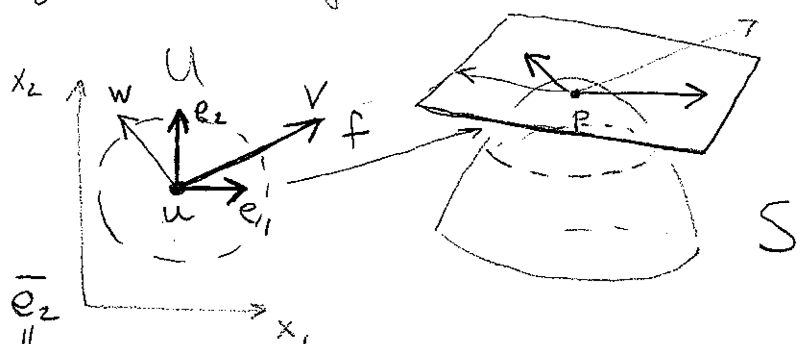
Def: $S \subseteq \mathbb{R}^3$ a smooth surface. The first fundamental form of S at p is the fn $I_p : T_pS \times T_pS \to \mathbb{R}$ defined by

$$I_p(v,w) = \langle v, w \rangle$$

[In general, the first fund form is the collection of all such]

I_p is a symmetric bilinear form [Query]

In a coordinate patch:



$e_1 = (1,0)$

$e_2 = (0,1)$

$$g_{ij}(u) = I_p(\underbrace{D_uf(e_i)}_{\bar{e}_1}, \underbrace{D_uf(e_j)}_{\bar{e}_2})$$

$g_{ii} = \text{length}\,(D_uf(e_i))^2$

$V = (v_1, v_2) = v_1e_1 + v_2e_2$

$W = (w_1, w_2) = w_1e_1 + w_2e_2$

$$I_p(v_1\bar{e}_1 + v_2\bar{e}_2,\, w_1\bar{e}_1 + w_2\bar{e}_2)$$

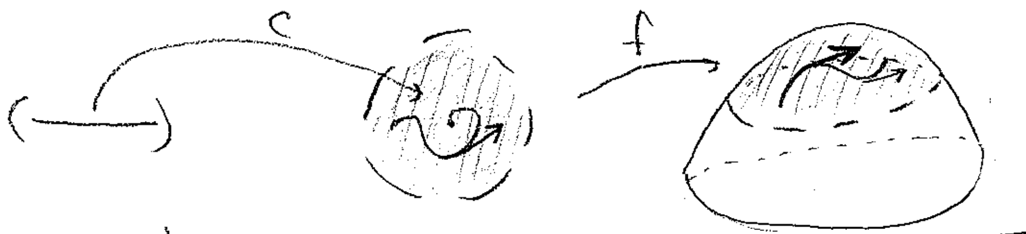
$$I_p(D_uf(v), D_uf(w)) = v_1w_1g_{11} + v_1w_2g_{12} + v_2w_1g_{21} + v_2w_2g_{12}.$$

$D_uf(v_1e_1 + v_2e_2) = v_1\bar{e}_1 + v_2\bar{e}_2$

$$\doteq V \underbrace{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}}_{\text{metric coeffs } G} W^T$$

Note: $g_{21} = g_{12}$ as I_p is symmetric.

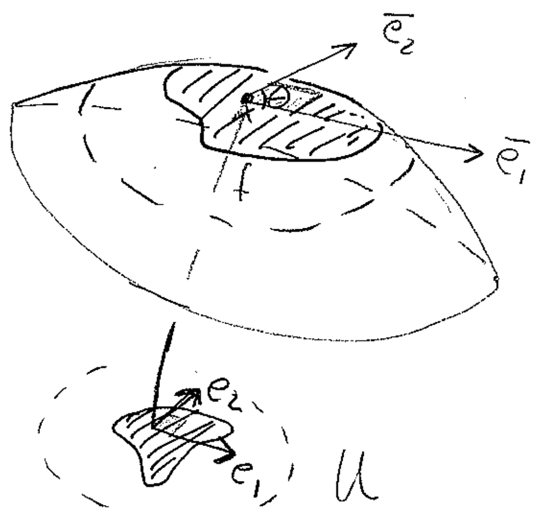
Ex: Suppose c is a curve in S, with image in a patch $f : U \to S$.



$$\text{Length}(f \circ c) = \int_a^b |(f\circ c)'(t)|\, dt = \int_a^b \sqrt{I_p\big(\underbrace{(f\circ c)'(t)}_{(D_{c(t)}f)(c'(t))}, (f\circ c)'(t)\big)}\, dt$$

$$= \int_a^b \sqrt{c'(t)\ G\ c'(t)^T}\, dt.$$

Area: For $A \subseteq f(U) \subseteq S$, the area of A is



$$\int_{f^{-1}(A)} |\bar{e}_1 \times \bar{e}_2|\, dx_1\, dx_2$$

$$= \int_{f^{-1}(A)} \sqrt{\det G}\ dx_1\, dx_2 \quad \text{(should it exist)}$$

as $\det G = g_{11} g_{22} - g_{12}^2 = |\bar{e}_1|^2 |\bar{e}_2|^2 - \overset{|\bar{e}_1|^2|\bar{e}_2|^2\cos^2\theta}{\overset{\shortparallel}{\langle \bar{e}_1, \bar{e}_2\rangle^2}}$

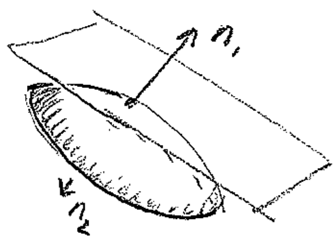
$= |\bar{e}_1|^2 |\bar{e}_2|^2 (1 - \cos^2\theta) = |\bar{e}_1 \times \bar{e}_2|$

Lemma: Area does not depend on which chart you use.

Note: Larger sets can be broken into pieces lying in charts in order to compute the area.

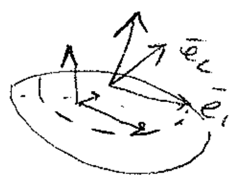
---

normal vectors:



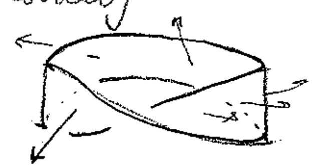
a normal vector to S at p is one $\perp$ to T_pS. [Usually look at unit unit normal vectors]

Over a coor patch can make the choice consistently:



$$n = \frac{\bar{e}_1 \times \bar{e}_2}{|\bar{e}_1 \times \bar{e}_2|}$$

may or may not be able to do so globally.

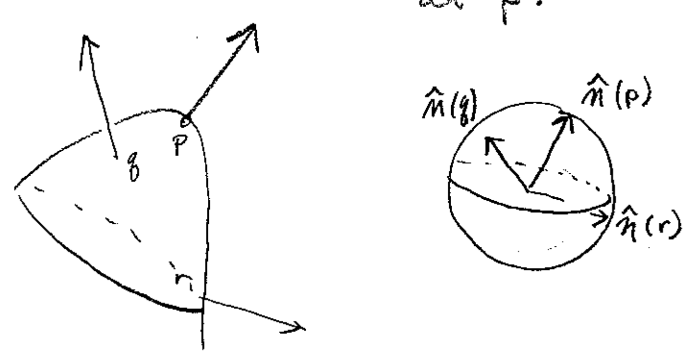


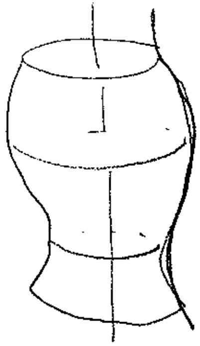
Gauss Map: [tool to define curvature.]

S a surface w/ a consistent choice of unit normal at each pt.

Then set $\hat{n}: S \longrightarrow S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$

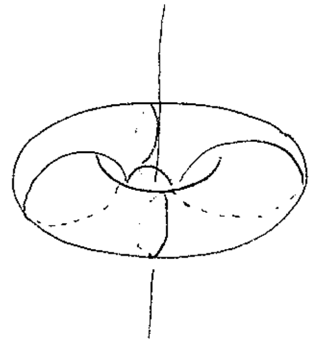
via $p \longmapsto$ unit normal at p.



Ex: 

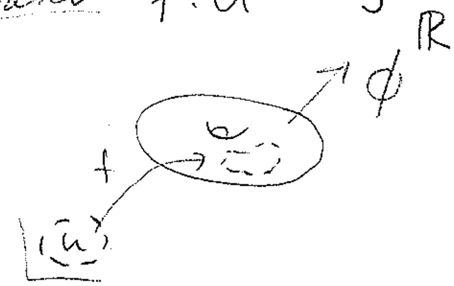
surface of revolution

[curve spun must have nowhere 0 velocity vector.]



What does it mean for $\phi : S \to \mathbb{R}$ to be smooth? or $\phi : S_1 \to S_2$?

<u>Def</u>: $\phi : S \to \mathbb{R}$ is smooth if for every <u>coordinate patch</u> $f : U \to S$ we have $\phi \circ f : U \to \mathbb{R}$ is smooth.



<u>Ex</u>: If W is an open set $\supseteq S$, $\phi : W \to \mathbb{R}$ is smooth, then $\phi : S \to \mathbb{R}$ is also smooth.

<u>Note</u>: It suffices to check this cond for a collection of coordinate charts that cover S because of.

<u>Change of Coordinates Lemma</u>: $S \subseteq \mathbb{R}^3$ a surface.

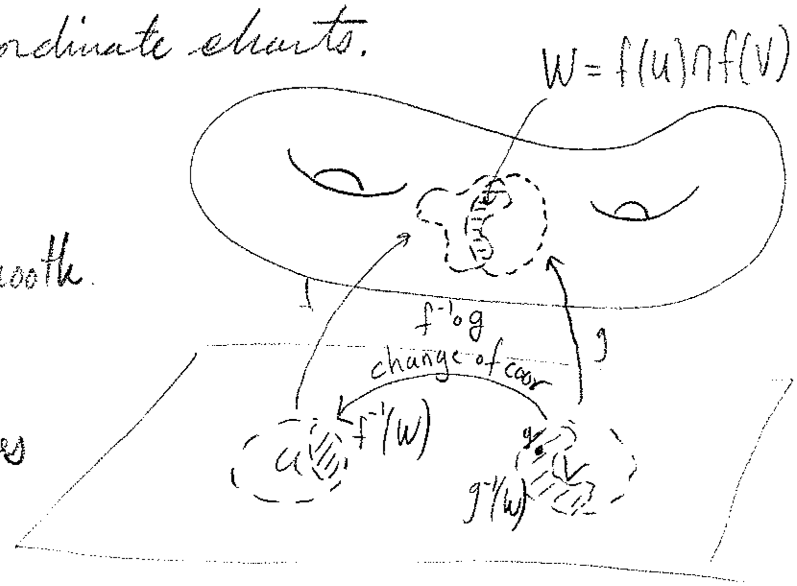
Let $f : U \to S$, $g : V \to S$ coordinate charts.

Set $W = f(U) \cap f(V)$. Then

$f^{-1} \circ g : g^{-1}(W) \to f(W)$ is smooth.

Is $\phi \circ g$ diff at $y \in f^{-1}(W)$? Yes as

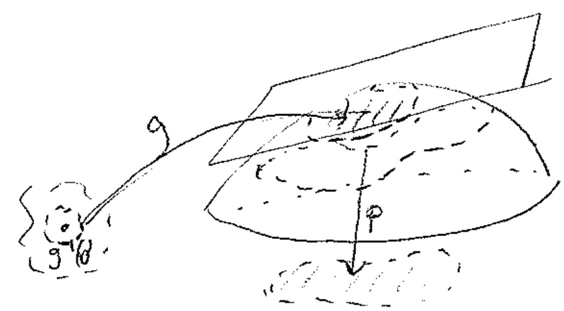
$\phi \circ g = (\phi \circ f) \circ (f^{-1} \circ g)$



Lemma: Let S be a smooth surface in $\mathbb{R}^3$, given $p \in S$, we can permute the coordinates so that there is a coor patch $f: \mathcal{U} \to S$ w/ $f(\mathcal{U}) \ni p$ of the form $f(x,y) = (x, y, h(x))$.

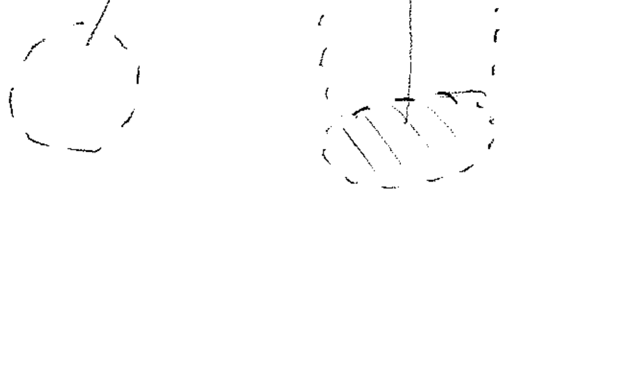
Pf: Take some coor patch $g: \mathcal{U} \to S$ containing p. Let $T = \text{image}(D_{g^{-1}(p)}\, g)$

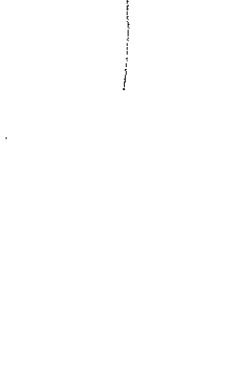
By permuting the vars, can assume the proj $P: \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$ is surjective when restricted to T.



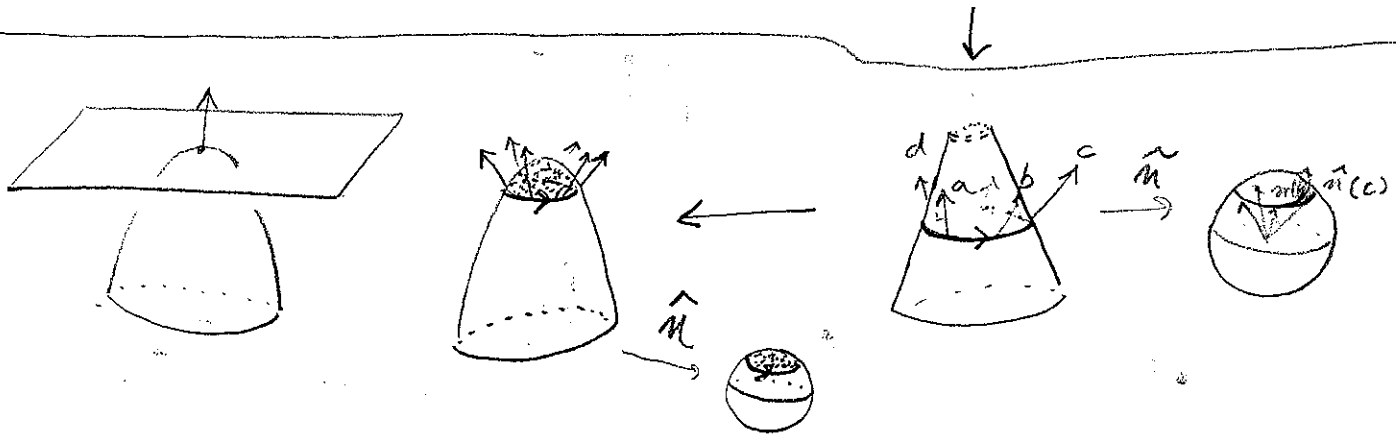
Then by the inverse function thm, $P \circ g$ is a diffeo when restricted to some nbhd W of $g^{-1}(p)$. Take $f = g \circ (P \circ g)^{-1}$.

Why does it imply the change of coor lemma?

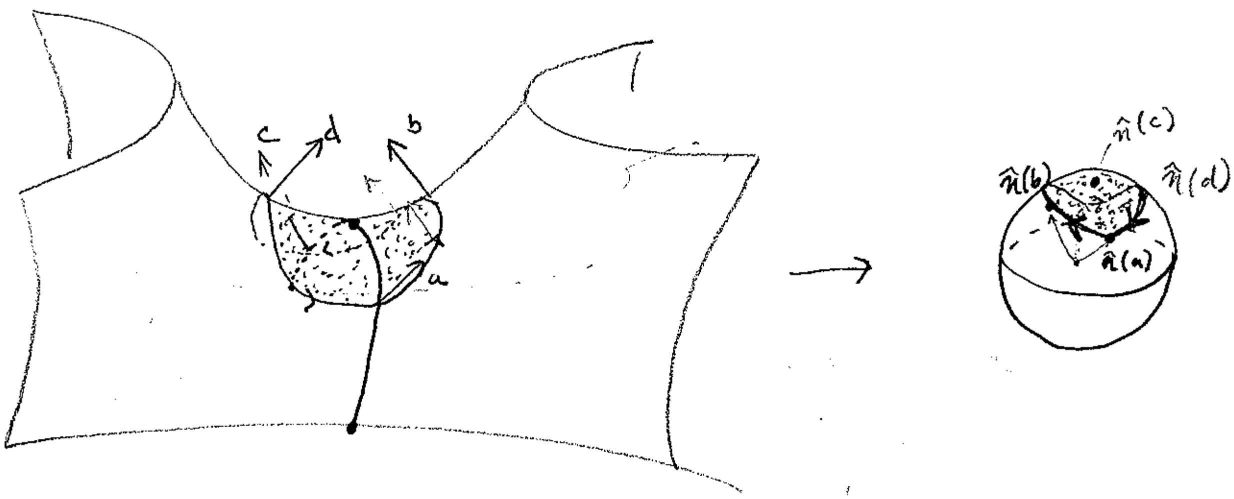




Ex:



didn't get to.



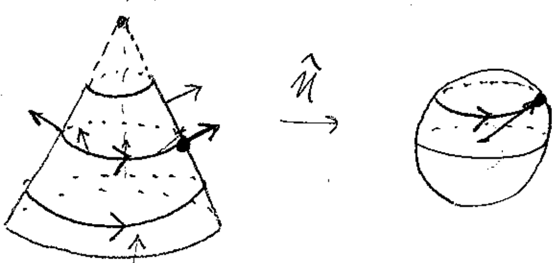
Lecture 10: Today: Curvature 101.

Last time: $S \subseteq \mathbb{R}^3$ smooth surface, with consistent unit normal

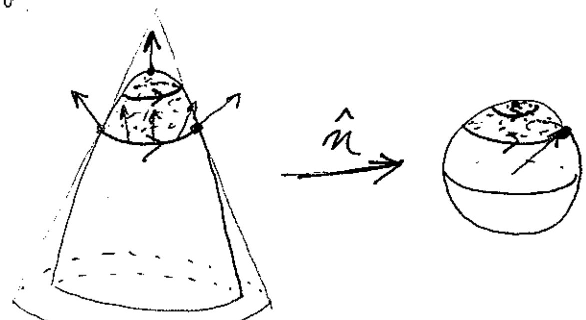
Gauss map: $\hat{n} : S \longrightarrow S^2 = \{ x \in \mathbb{R}^3 \mid |x| = 1 \}$

$$p \longmapsto \text{unit normal at } p.$$

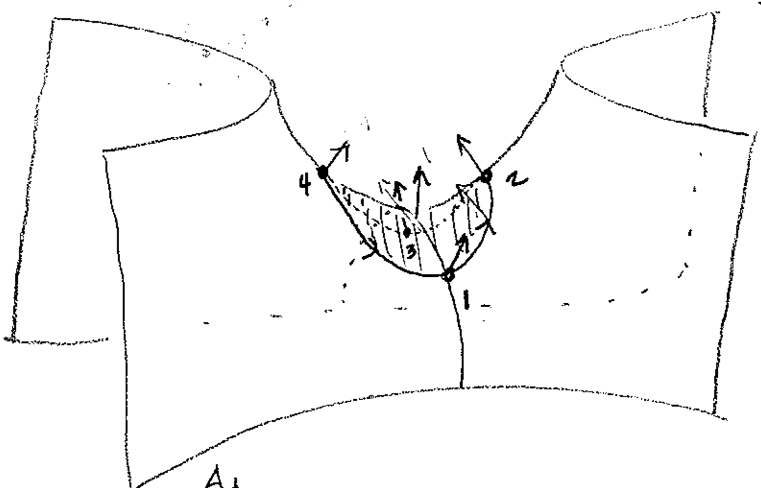
[Examples from last time, plane, cylinder, sphere, ...]

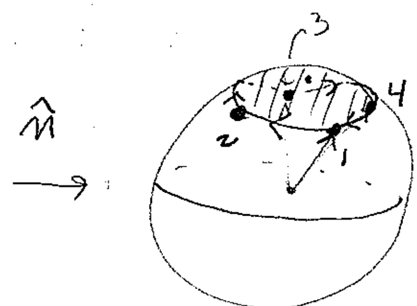


what is the image here.

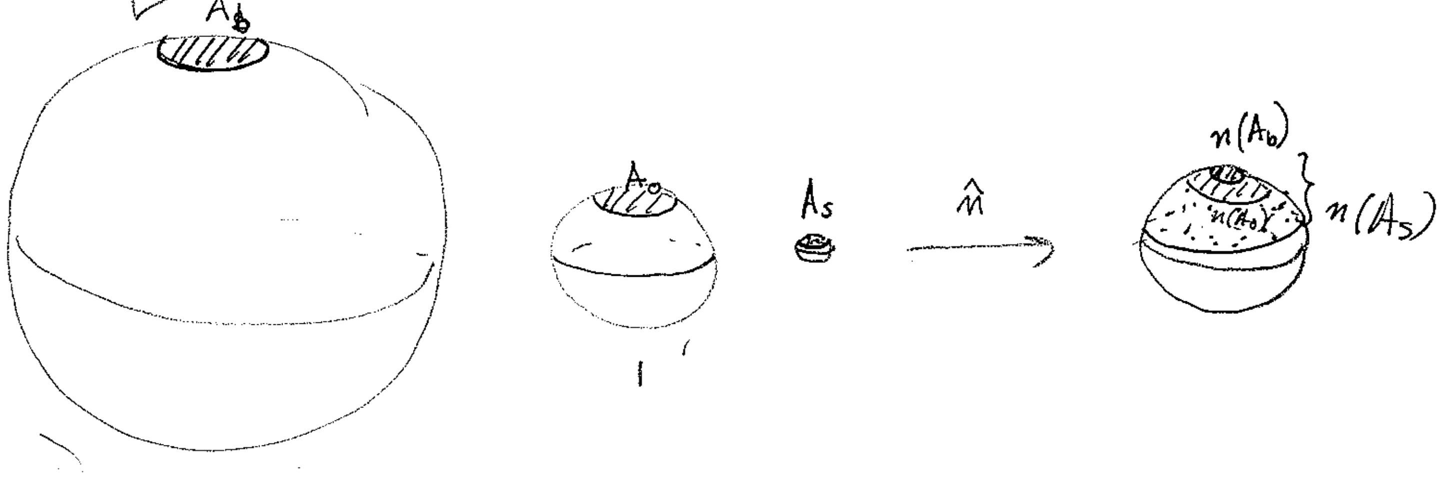


tangent spaces agree along





"turned over like a pancake."

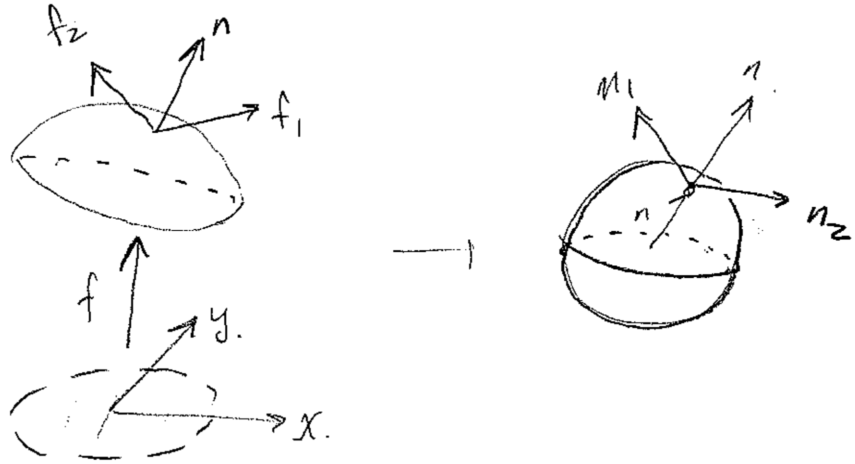


Gauss curvature: $K(p) = \lim\limits_{A \to \{p\}} \dfrac{\text{Area}_0(\hat{n}(A))}{\text{Area}(A)}$

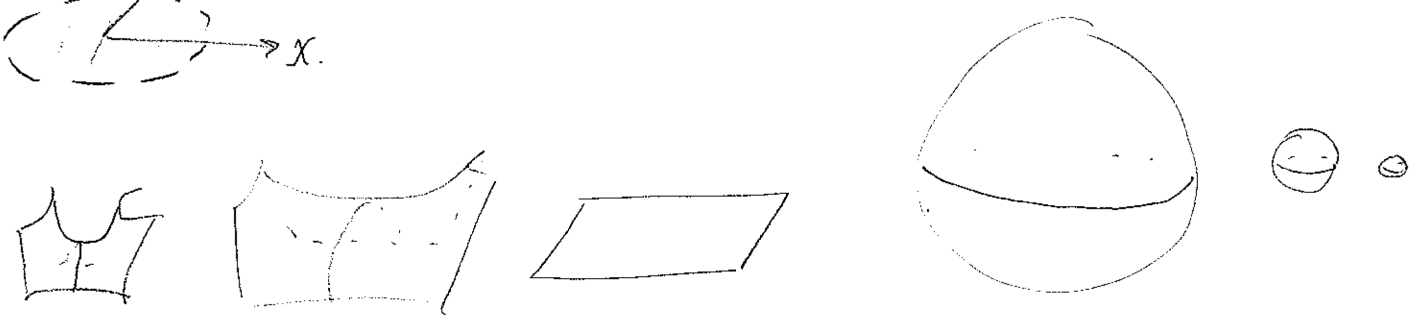
where $A \subseteq f(U)$ a coordinate patch

$$\text{Area}(A) = \int_{f^{-1}(A)} \langle f_1 \times f_2 , n \rangle \, dx \, dy \qquad f_1 = \frac{\partial f}{\partial x} = \bar{e}_1 \quad f_2 = \frac{\partial f}{\partial y} = \bar{e}_2$$

$$\text{Area}_0(A) \int_{f^{-1}(A)} \langle n_1 \times n_2 , n \rangle \, dx \, dy \qquad n_1 = \frac{\partial n}{\partial x} \quad n_2 = \frac{\partial n}{\partial y}.$$



$D_p \hat{n}(f_i) = n_i$



$K = 0$

$-\infty$ $+\infty$

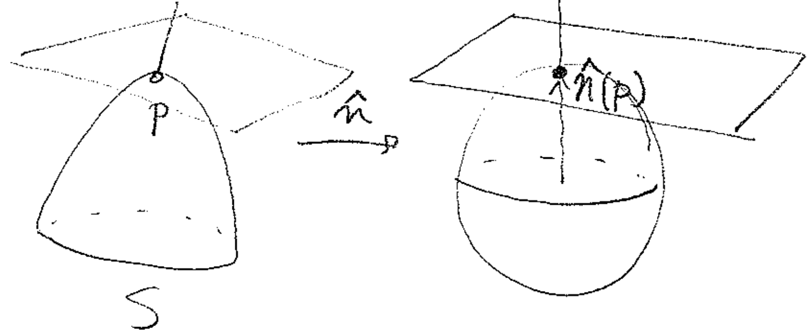
Dilating the surface $S \longrightarrow rS$ changes $K(rS, rp)$

$= \frac{1}{r^2} K(S, p)$.

Problem: do this well def? How do we compute?? [Note intr. nature]

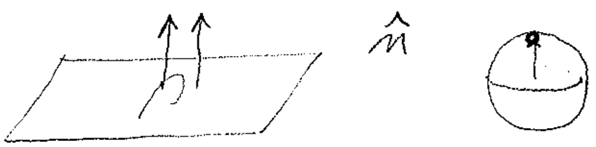
Alternate approach:

Weingarten map:



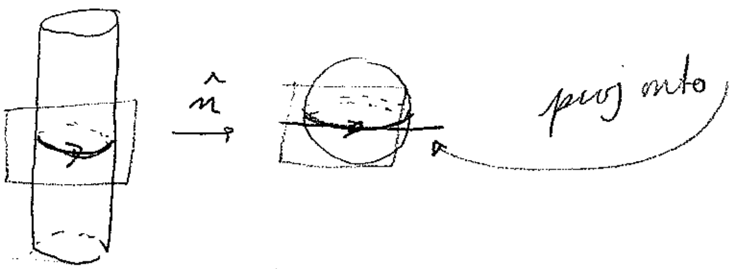
$$D_p \hat{n} : T_p S \to T_{\hat{n}(p)} S^2.$$

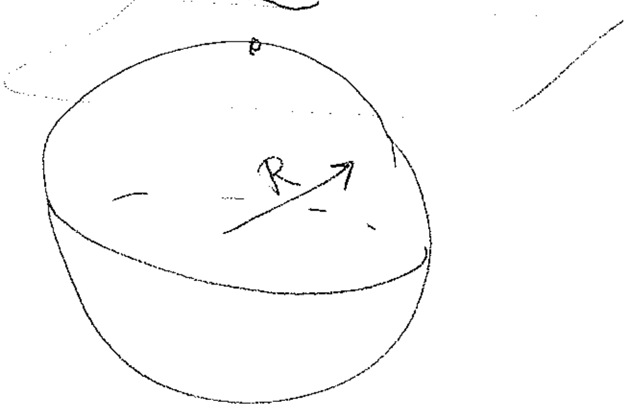
Note: These are the same plane! If we identify them, get $L : T_p S \to T_p S$ a linear map.

Ex: 

$L = 0$

Note extrinsic nature.





S^2_R

$$L = \frac{1}{R} I.$$

[Why did I emph range of L is the same as the domain]

$L : V \longrightarrow W$ doesn't have many invariants.

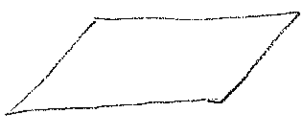
$L : V \longrightarrow V$ has $\det V$ and $\operatorname{tr} V$

Def: Gaussian curvature at p, $K(p) = \det L$

Mean curvature at p, $H(p) = \frac{1}{2} \operatorname{tr} L$

[explain why this makes sense rel our earlier discussion]

Ex:

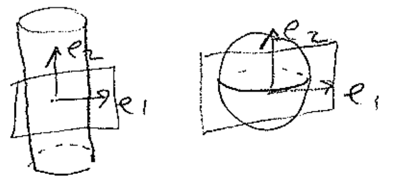


$L = 0 \qquad K = 0, \qquad H = 0$

differ!

intrinsic

extrinsic



$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad K = 0 \qquad H = \frac{1}{2}$

$K = \frac{1}{R^2} \qquad H = \frac{2}{R}$

What is mean curve good for?

Also did isometries.

Def: A surface is *minimal* if $H(p) = 0 \ \forall p$.

Ex: Soap bubble surface.

Existence: Plateau's problem / Jesse Douglas, Tibor Rado 1930

Lecture 11: Last time: $\hat{n} : S \to S^2$ Gauss map

Midterm handed out on Wed. Open notes, book.

$L = D_p\hat{n} : T_pS \to T_pS$ Weingarten map.

Gaussian curvature: $K(p) = \det L$ [diff. def of area]

Mean curvature: $H(p) = \frac{1}{2}\operatorname{tr} L$

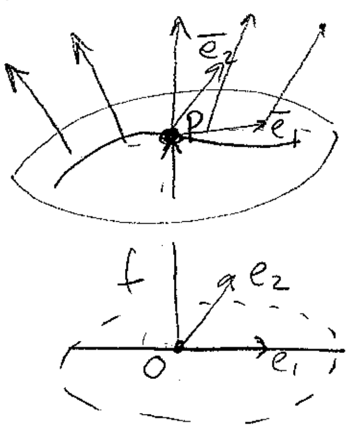
Today: more about L, and how K, H relate to curvature of curves.

Lemma: L is self-adjoint, i.e. $\langle Lv, w\rangle = \langle v, Lw\rangle$

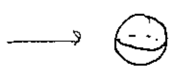
Equiv, the matrix of L w.r.t. an orthonormal basis is symmetric.

[e_1, e_2 then $L_{ij} = \langle Le_j, e_i\rangle$]

Pf: Let $f : U \to S$ be a chart w/ $f(0) = p$. May assume v and w are linearly indep and



$v = \bar{e}_1 = D_0 f(e_1) = \dfrac{\partial f}{\partial x}$ $\quad w = \bar{e}_2$



Set $N = \hat{n} \circ f : U \to S^2$

Note $L(v) = D_p\hat{n}(v) = D_0 N(e_1) = \dfrac{\partial N}{\partial x}$

$L(w) = \dfrac{\partial N}{\partial y}$ $\quad =$

Note: $\left\langle \dfrac{\partial f}{\partial x}, N \right\rangle \left[= 0\right]$ on U. Query $\Longrightarrow \left\langle \dfrac{\partial f}{\partial y \partial x}, N \right\rangle + \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial N}{\partial y} \right\rangle = 0$

$\left\langle \dfrac{\partial f}{\partial y}, N \right\rangle = 0 \Longrightarrow \left\langle \dfrac{\partial f}{\partial x \partial y}, N \right\rangle + \left\langle \dfrac{\partial f}{\partial y}, \dfrac{\partial N}{\partial x} \right\rangle = 0$

$\Longrightarrow$ at p $\quad \langle v, L(w)\rangle = \langle w, L(v)\rangle$. ∎

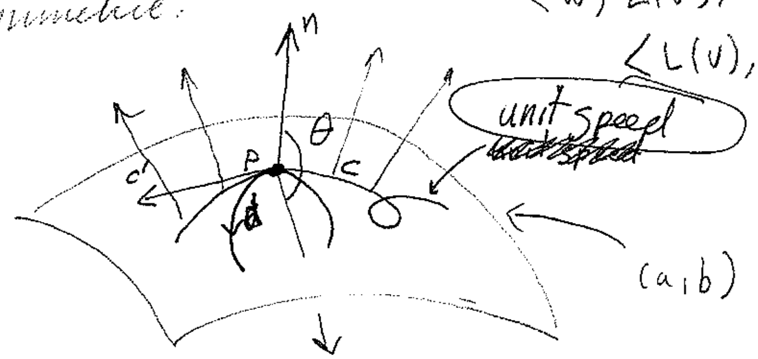
Def: The 2^{nd} fundamental form at p is def by

$$\mathrm{II}_p : T_pS \longrightarrow T_p S$$
$$(v,w) \longmapsto \langle L(v), \underline{w} \rangle$$

$\mathrm{II}_p(v, w_1 + w_2)$

$$\mathrm{II}_p(W,V) = \langle L(w), v\rangle = \langle w, L(v)\rangle = \langle L(v), w\rangle$$

Note: II_p is bilinear and symmetric:



What does II_p measure??

normal curvature of c at p:

$$K_n = -K\,\langle N_c, n\rangle = -K \cos\theta. \qquad N_c - \text{curve normal}$$

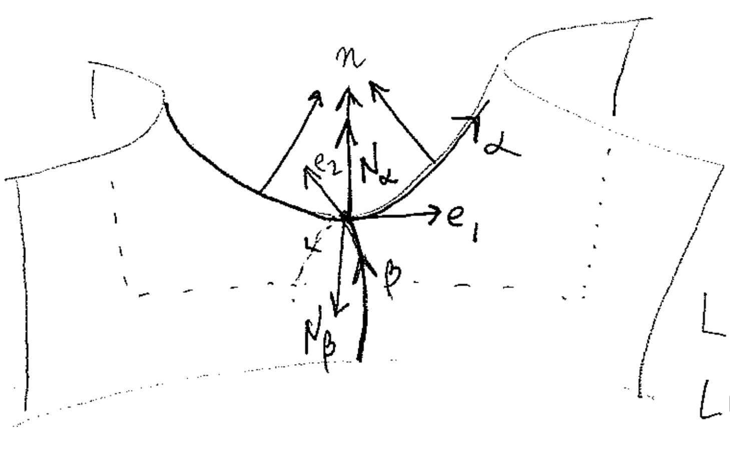
curvature of c at p [Measures external curvature.]

Thm: $\mathrm{II}_p(c', c') = K_n$. [Note: only depends on c'!]

Pf: Assume c is unit speed. Let $n(t)$ be the normal to S at $c(t)$

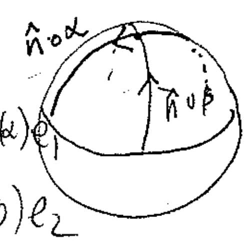
As $\langle n(t), c'(t)\rangle = 0$ we have

$$\langle n'(t), c'(t)\rangle = -\langle n(t), c''(t)\rangle = -\langle n(t), K(t) N_c(t)\rangle = K_n$$
$$\langle D_p \hat{n}(c'(t)), c'(t)\rangle = \mathrm{II}_p(c', c') \qquad \blacksquare$$



$$\mathrm{II}_p(e_1, e_1) = -K(\alpha)$$

$$\mathrm{II}_p(e_2, e_2) = -K(\beta)$$



$$L = \begin{pmatrix} -K(\alpha) & 0 \\ 0 & -K(\beta) \end{pmatrix}$$

$$L(e_1) = -K(\alpha) e_1$$
$$L(e_2) = K(\beta) e_2$$

Hence: $$K(p) = -K(\alpha)K(\beta)$$

$$H(p) = \frac{1}{2}(-K(\alpha) + K(\beta))$$

In general as L is symmetric, $\exists$ an orthonormal basis e_1, e_2 of T_pS so that L is diagonal $L = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ $\underbrace{K_1 \geq K_2}_{\text{principal curvatures.}}$

For any unit vector $v = \sin\theta\, e_1 + \cos\theta\, e_2$

we have

$$\mathrm{II}_p(v,v) = \langle L(v), v \rangle = \langle K_1 \sin\theta\, e_1 + K_2 \cos\theta\, e_2, v \rangle$$

$$= K_1 \sin^2\theta + K_2 \cos^2\theta$$

Geometrically:

K_1 = max normal curve over all curves $c \subseteq S$ passing through p.

K_2 = min ...

Note: Because there are two choices for N, L is only defined up to sign, so is H, K_1, K_2

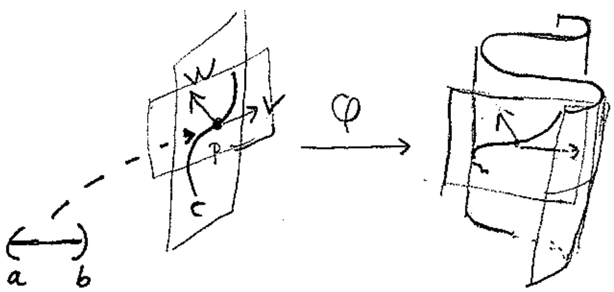
K is well defined, regardless.

Lecture 12: Intrinsic vs. Extrinsic.

Def: $\varphi : S_1 \to S_2$ be a smooth map of surfaces in $\mathbb{R}^3$.
Then φ is a <u>local isometry</u> if $\forall p$ and $v, w \in T_p S_1$

we have

$$I_{S_1,p}(v,w) = I_{S_2,\varphi(p)}\left(D_p\varphi(v), D_p\varphi(w)\right).$$



φ is an <u>isometry</u> if it is <u>also</u> a diffeomorphism.

Def: A property is <u>intrinsic</u> if it is invariant under isometries.

<u>Intrinsic</u>: • Length of a curve

$$\mathrm{len}(c) = \mathrm{len}(\varphi \circ c)$$

$$\int_a^b \sqrt{I_{c(t)}(c', c')}\, dt \qquad \int_a^b \sqrt{I_{c\circ\varphi(t)}((\varphi\circ c)'(t), \text{same})}\, dt$$

$$(\varphi\circ c)'(t) = D_{c(t)}\varphi(c'(t))$$

• area [come back to this]

<u>Extrinsic</u>: • dist between points in $\mathbb{R}^3$.
• mean curvature.

<u>Notes</u>: • local isometries are local diffeomorphisms [Query?]
• φ is a local isom if $\forall$ charts $U \to S_1$

we have

$$\underbrace{g_{ij}^{S_1}}_{\text{metric coeffs for } I_{S_1}} = g_{ij}^{S_2} \Big\} \text{ metric coeffs for } U \xrightarrow{\varphi\circ f} S_2$$

[relate back to area.]

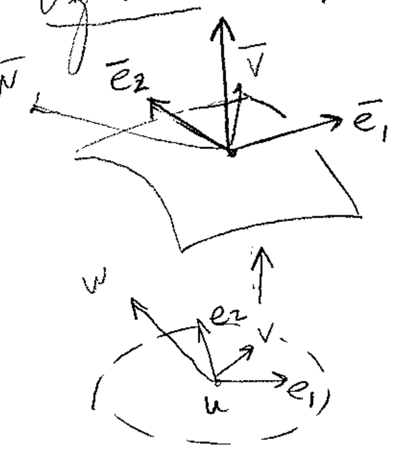
Theorema Egregium ("remarkable theorem")

$\varphi : S_1 \to S_2$ is a local isometry. Then $\forall\, p \in S_1$ we have

$$K_{S_1}(p) = K_{S_2}(\varphi(p))$$

[i.e. Gaussian curvature is intrinsic.] [Explain why it is surprising.]

Pf idea: express K in terms of g_{ij} and its derivatives.



$$\bar{e}_1 = Df(e_1) = \frac{\partial f}{\partial x} = f_x \qquad \bar{e}_2 = f_y$$

$$g_{ij}(u) = I_p(\bar{e}_i, \bar{e}_j) \qquad g_{12} = g_{21}$$

$$l_{ij}(u) = II_p(\bar{e}_i, \bar{e}_j) \qquad l_{12} = l_{21}$$

(L_{ij}) = matrix of Weingarten map w.r.t. $\{\bar{e}_1, \bar{e}_2\}$

[Focus on this as it is symmetric, whereas typically isn't.]

column vectors

v, w in terms of e_1, e_2 $\quad I_p(\bar{v}, \bar{w}) = v^T (g_{ij}) w$

$$
\begin{aligned}
II_p(\bar{v}, \bar{w}) &= v^T (l_{ij}) w \\
&= ((L_{ij}) v)^T (g_{ij}) w \\
&= v^T (L_{ij})^T g_{ij} w \implies (l_{ij}) = (L_{ij})^T g_{ij}
\end{aligned}
$$

taking transpose

$$\Rightarrow (l_{ij}) = (g_{ij})(L_{ij}) \Rightarrow (L_{ij}) = (g_{ij})^{-1}(l_{ij})$$

$$\Rightarrow K = \det(L_{ij}) = \frac{\det(l_{ij})}{\det(g_{ij})} = \frac{l_{11} l_{22} - l_{12}^2}{g_{11} g_{22} - g_{12}^2}$$

need can be expressed in terms of g_{ij}

$$\begin{pmatrix} l_{12} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$$

Take the normal $n = \dfrac{\bar{e}_1 \times \bar{e}_2}{|\bar{e}_1 \times \bar{e}_2|} : U \longrightarrow S^2$

$= \hat{n} \circ f$

basis for $\mathbb{R}^3 = (f_x, f_y, n)$

$$f_{xx} = \Gamma_{11}^{1} f_x + \Gamma_{11}^{2} f_y - l_{11} n$$

lemma: $\langle f_x, n \rangle = 0 \Rightarrow$

$\langle f_{xx}, n \rangle = -\langle f_x, n_x \rangle$

$\langle f_x, n_x \rangle = L(f_x)$

$$f_{xy} = \Gamma_{12}^{1} f_x + \Gamma_{12}^{2} f_y - l_{12} n$$

$$f_{yy} = \underbrace{\Gamma_{22}^{1} f_x + \Gamma_{22}^{2} f_y}_{} - l_{22} n$$

definition: Γ_{jk}^{i} – Christoffel symbol

$$\frac{1}{2}(g_{11})_x = \langle f_{xx}, f_x \rangle = \Gamma_{11}^{1} g_{11} + \Gamma_{11}^{2} g_{12}$$

$$(g_{12})_x - \frac{1}{2}(g_{11})_y = \langle f_{xx}, f_y \rangle = \Gamma_{11}^{1} g_{12} + \Gamma_{11}^{2} g_{22}$$

$\Rightarrow$ have linear system w/ det $g_{11}g_{22} - g_{12}^2 > 0$

$|\bar{e}_1|^2 |\bar{e}_2|^2 - |\langle \bar{e}_1, \bar{e}_2 \rangle|^2$

for Γ_{11}^{1}, Γ_{11}^{2} in terms of derivatives of g_{ij}

Same for rest $\Rightarrow \Gamma_{jk}^{i}$ are determined by g_{ij}.

A calculation then shows:

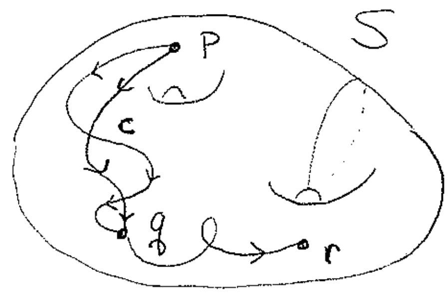
$$l_{11} l_{22} - l_{12}^2 = \sum_{r=1}^{2} g_{1r} \left(\frac{\partial \Gamma_{22}^{r}}{\partial x} - \frac{\partial \Gamma_{21}^{r}}{\partial y} + \sum_{m=1}^{2} \left(\Gamma_{22}^{m} \Gamma_{m1}^{k} - \Gamma_{m1}^{r} \Gamma_{m2}^{r} \right) \right)$$

$\Rightarrow$ Theorema Egregium.

Lecture 13: Last time: Intrinsic v. Extrinsic

Today: Geodesics and distances.

Intrinsic Distance:

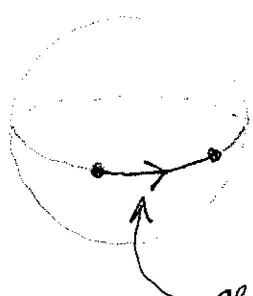


$$d(p,q) = \inf \{ \operatorname{len}(c) \mid \text{a path in } S \text{ joining } p \text{ to } q \}$$

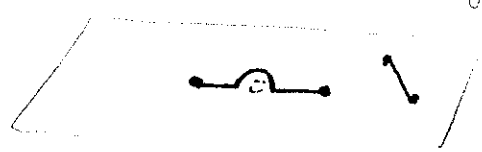
Ex: This makes S into a metric space

Fact: Provided S is closed in $\mathbb{R}^3$, $d(p,q) = \operatorname{len}(c)$ for some particular c.

$S = \mathbb{R}^2 \setminus \{0\}$ $\qquad d(p,q) = $ Euc. dist



vs.



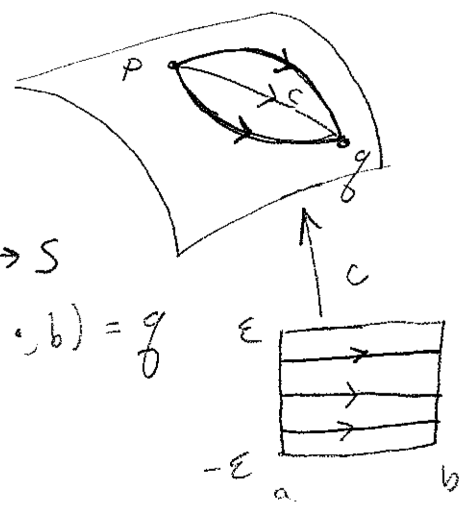
geodesics are such length minimizing paths.

Variational Characterization:

$\begin{bmatrix}\text{unit}\\ \text{speed}\end{bmatrix}$ $c : [a,b] \longrightarrow S$

$C : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow S$

w/ $C(\cdot, a) = p \quad C(\cdot, b) = q$



$c_\alpha : [a,b] \longrightarrow S$

$c_\alpha(t) = C(\alpha, t)$

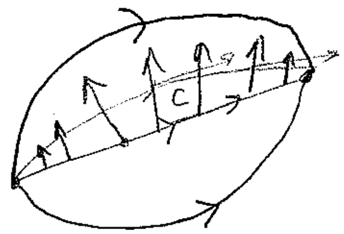
and $c_0 = c$

Talk about expectation

If c is a geodesic, then $\operatorname{len}(c) \leq \operatorname{len}(c_\alpha)$ for all α.

Hence

$$0 = \frac{d\,\mathrm{len}(c_\alpha)}{d\alpha}\bigg|_{\alpha=0} = \frac{\partial}{\partial\alpha}\left(\int_a^b \sqrt{\langle \underbrace{c_\alpha'(t)}_{=\frac{\partial c}{\partial t}(t,\alpha)}, c_\alpha'(t)\rangle}\, dt\right)\bigg|_{\alpha=0}$$



$$= \int_a^b \left(\frac{1/2}{\sqrt{\langle\,,\,\rangle}}\cdot 2\left\langle \frac{\partial c}{\partial\alpha\,\partial t}(\alpha,t), \frac{\partial c}{\partial t}(\alpha,t)\right\rangle\right)\bigg|_{\alpha=0} dt$$

$$= \int_a^b \langle V'(t), c'(t)\rangle\, dt$$

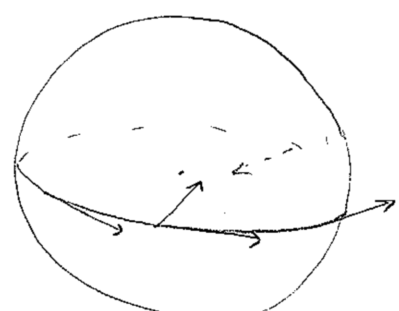
$V(t) = \frac{\partial c}{\partial\alpha}(0,t)$

$$= \underbrace{\langle V(b), c'(b)\rangle}_{=0} - \underbrace{\langle V(a), c'(a)\rangle}_{=0} - \int \langle V(t), c''(t)\rangle dt$$

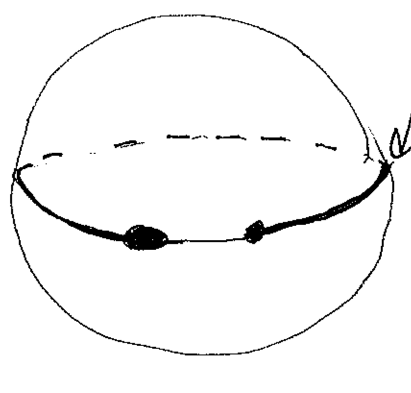
$$= -\int \langle V(t), c''(t)\rangle\, dt$$

So, c a geodesic $\Rightarrow$ $c''(t)$ is <u>normal</u> to S for all t.

[<u>Actually:</u>]



<u>Def</u>: A <u>geodesic</u> in S is a curve $c:(a,b)\to S$ such that $c''(t)$ is normal to S for all t.

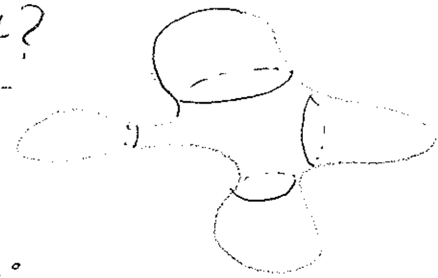


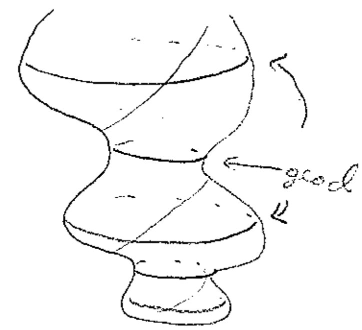
also a geodesic

<u>Note</u>, need not minimize length.

<u>Note</u>: is <u>intrinsic</u>, by var. char.

Do geodesics always exist?

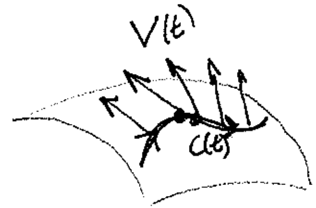




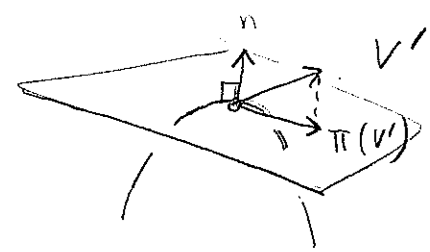
Covariant differentiation:

$c : (a,b) \longrightarrow S$ a curve

$V : (a,b) \longrightarrow \mathbb{R}^3$ a vector field along c, i.e. $V(t) \in T_{c(t)}S$



$$\frac{DV}{dt}(t) = \text{Projection onto } T_{c(t)} \text{ of } V'$$

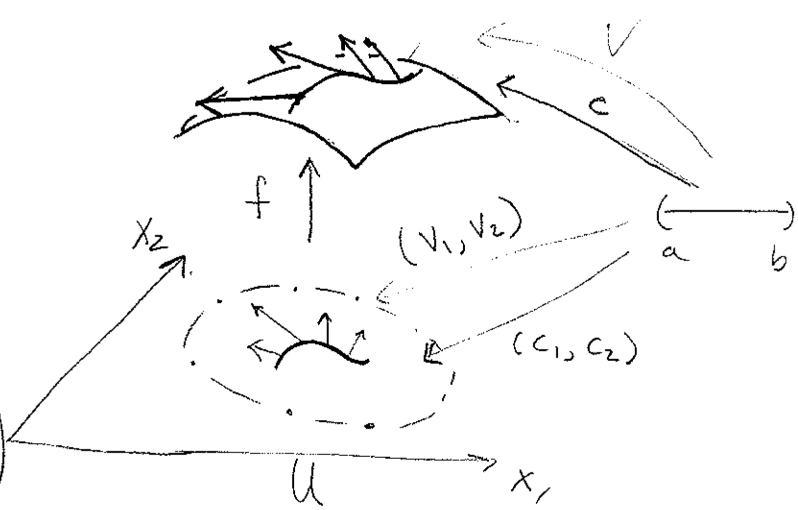


Note: c is a geodesic iff $\frac{Dc'}{dt} = 0$ for all t.

In local coors.

$$V(t) = v_1(t)\, f_{x_1}(c_1(t), c_2(t)) + v_2(t)\, f_{x_2}(c_1(t), c_2(t))$$

$$V' = \sum_{i=1}^{2} \left(\underline{\underline{v_i'}}\, f_{x_i} + \underline{\underline{v_i}}\, f_{x_i x_1}\, \underline{\underline{c_1'}} + \underline{\underline{v_i}}\, f_{x_i x_2}\, \underline{\underline{c_2'}} \right)$$



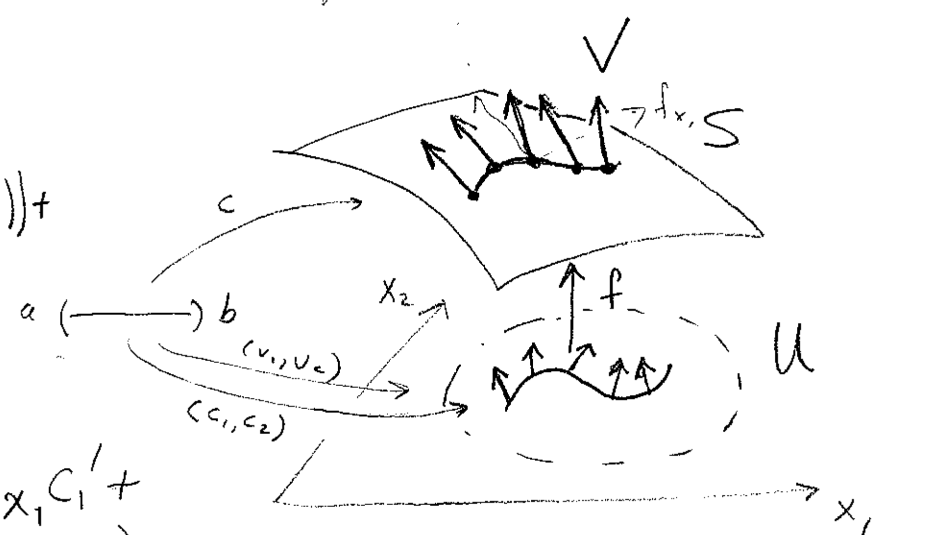
$$f_{x_i x_j} = \Gamma_{ij}^{1} f_{x_1} + \Gamma_{ij}^{2} f_{x_2} - l_{ij}\, n$$ throw away

$$\frac{DV}{dt} = \sum_{i=1}^{2} \left(v_i' + \sum_{j,k=1}^{2} \Gamma_{jk}^{i}\, v_j\, c_k' \right) \underline{\underline{f_{x_i}}} \Rightarrow \frac{DV}{dt} \text{ is intrinsic.}$$

Lecture 14: Last time: Copy covariant diff from prev. page.

In local coordinates:

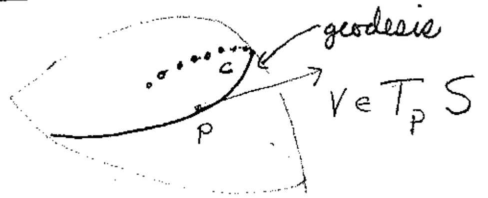
$$V(t) = v_1(t) f_{x_1}(c_1(t), c_2(t)) + v_2(t) f_{x_2}(c_1(t), c_2(t))$$



$$V' = \sum_{i=1}^{2} \left(v_i' f_{x_i} + v_i f_{x_i x_1} c_1' + v_i f_{x_i x_2} c_2' \right)$$

$$f_{x_i x_j} = \Gamma_{ij}^{1} f_{x_1} + \Gamma_{ij}^{2} f_{x_2} \underbrace{- l_{ij} n}_{\text{ignore}}$$

$$\frac{DV}{dt} = \sum_{i=1}^{2} f_{x_i} \left(v_i' + \sum_{j,k=1}^{2} v_j \Gamma_{jk}^{i} c_k' \right) \Rightarrow \boxed{\frac{dV}{dt} \text{ is intrinsk}}$$



Existence of geodesics:

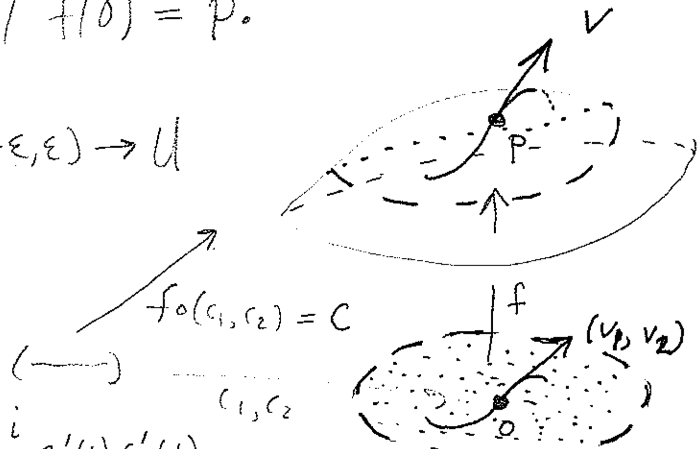
Thm: $S \subseteq \mathbb{R}^3$ a smooth surface. Let $v \in T_p S$. $\exists\ \varepsilon > 0$ and a geodesic $c : (-\varepsilon, \varepsilon) \to S$ s.t. $c(0) = p$ and $c'(0) = v$.

Moreover, if $\tilde{c} : (-\delta, \delta) \to S$ is another such geod, then $c = \tilde{c}$ on $(-\varepsilon, \varepsilon) \cap (-\delta, \delta)$.

[Note: geodesics are always constant speed.]

Pf: Choose coordinates $f: U \to S$ w/ $f(0) = p$.

Consider a curve in U, $(c_1, c_2): (-\varepsilon, \varepsilon) \to U$



$$c'(t) = c_1'(t) f_{x_1} + c_2'(t) f_{x_2} + ?\,n$$

$$\frac{Dc'}{dt} = 0 \iff c_i'' = -\sum_{j,k=1}^{2} \Gamma_{jk}^i c_j'(t) c_k'(t)$$

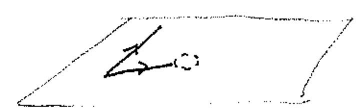
Take $d_i = c_i'$, get a 1^{st} order system

$$d_i = c_i', \quad d_i' = -\sum_{j,k=1}^{2} \Gamma_{jk}^i d_j d_k \quad \text{in } (c_i, d_i)$$

By math 2a, there exist a unique solution to these equations with initial cond $c_1(0) = c_2(0) = 0 \quad d_1(0) = v_1 \quad d_2(0) = v_2$ ∎

where $V = v_1 f_{x_1} + v_2 f_{x_2}$.

__Note__: geod may not exist for all time.



Def: A __symmetry__ of S is an isometry $\varphi: S \to S$.

__Cor__: Suppose φ is a symmetry of S which fixes p in S and $V \in T_pS$. Then the geodesic through p w/ tangent vector V is pointwise fixed by φ.

Pf: Let c be the specified geod. Then $\varphi \circ c$ is also a geod
as geod are intrinsic.

[Skip and put on HW; replace w/ quick discussion of some examples.]

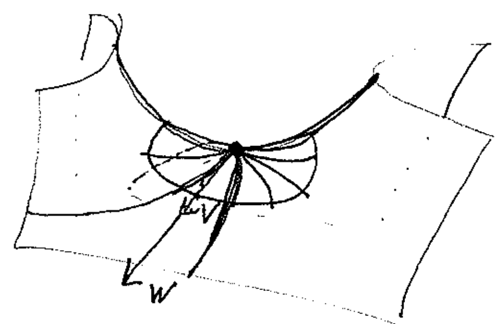
and if $c(0) = p$ then $\varphi \circ c(0) = \varphi(p) = p$

$c'(0) = v$ $\qquad (\varphi \circ c)'(0) = (D_p\varphi)\, c'(0) = v.$ $\Rightarrow c = \varphi \circ c$ $\blacksquare$

Thus: • great circles are the sphere are (all) geod.

• ellipsoid $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$

• surface of revolution



Exponential Map:

$\wedge v \in T_pS$. Set

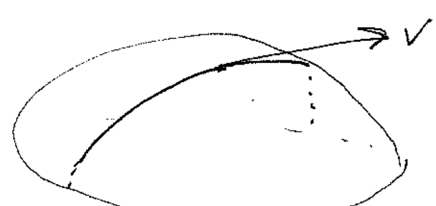
$$\rho_v = \sup\{r \in \mathbb{R}^+ \mid \exists \text{ a geod } c : (-r, r) \to S \text{ w/ } c(0) = p,\ c'(0) = v\}$$

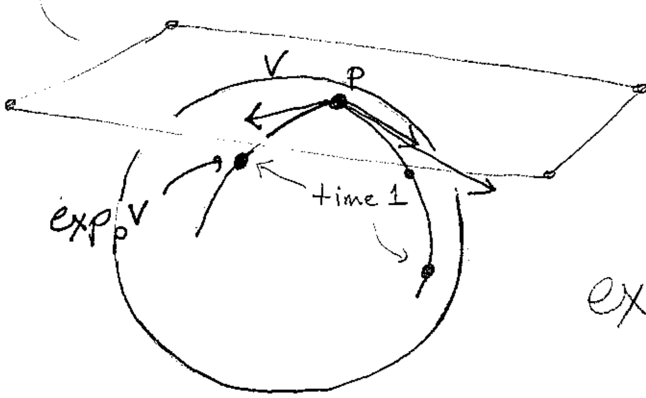
[pos $\rho_v = \infty$]

move

Note: • $\rho_v > 0$. • $\exists$ a geod $c : (-\rho_v, \rho_v) \to S$ w/ $c'(0) = v$

• $s \in \mathbb{R} \setminus \{0\}$ then $\rho_{sv} = \frac{\rho_v}{|s|}$.





$$E_p = \{v \in T_pM \mid \rho_v > 1\}$$

$$\exp_p : E_p \to S$$

$v \mapsto c(1)$ where c is the geodesic such that $c(0) = p$ and $c'(0) = v$.

distance traveled equals $|v|$

Note: $\exp_p(s\overset{\text{unit vector}}{V}) : (-\rho_v, \rho_v) \longrightarrow S$ is the geod through p w/ tangent vector V.

Thm: For any p, $\exists$ an open set $0 \in U \subseteq E_p$ on which $\exp_p$ is smooth.

Pf: Math 2b.

Cor: $\exists\, U \ni 0$ in $T_p S$ such that $\exp_p\big|_U$ is a diffeomorphism.

Pf: $D_0(\exp_p) = Id$