Lecture 35: Projective space II.

Last time:

$$\mathbb{P}^n_k = \left\{ \begin{matrix} \text{all lines in} \\ k^{n+1} \text{ through } 0 \end{matrix} \right\} = \left\{ x \in k^{n+1} \mid x \neq 0 \right\} \Big/ \begin{matrix} x \sim \lambda x \\ \text{for } \lambda \in k^\times \end{matrix}$$

$$= \left\{ (x_1 : \cdots : x_n : 1) \right\} \cup \left\{ (x_1 : x_2 : \cdots : x_n : 0) \right\}$$

not all 0

$$= k^n \cup \mathbb{P}^{n-1}_k$$ [Example of a moduli space]

Focus on $\mathbb{P}^2_{\mathbb{R}}$ =  $/ x \sim -x$

Def: $f \in \mathbb{R}[x,y,z]$ is homogenous if each term $d\, x^a y^b z^c$ has the same total degree $a+b+c$.

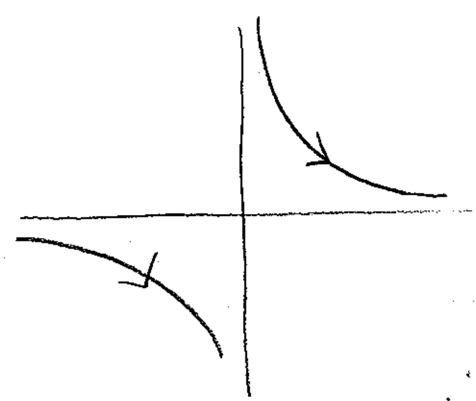
For such f, can define the projective variety

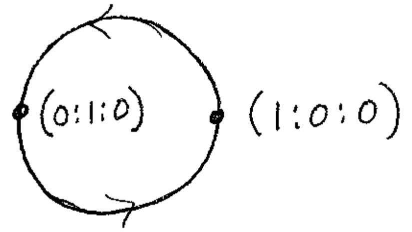
$$\mathbb{V}(f) = \left\{ (a:b:c) \in \mathbb{P}^2_{\mathbb{R}} \mid f(a,b,c) = 0 \right\}$$

Ex: $V = \mathbb{V}(xy - z^2)$

$V \cap \mathbb{R}^2 = \mathbb{V}_{\mathbb{R}^2}(xy-1)$

$V \cap \mathbb{P}^1_{\mathbb{R}} = \left\{ (1:0:0), (0:1:0) \right\}$





Will show that it is the "same" as $V_{\mathbb{R}^2}(x^2+y^2-1)$.

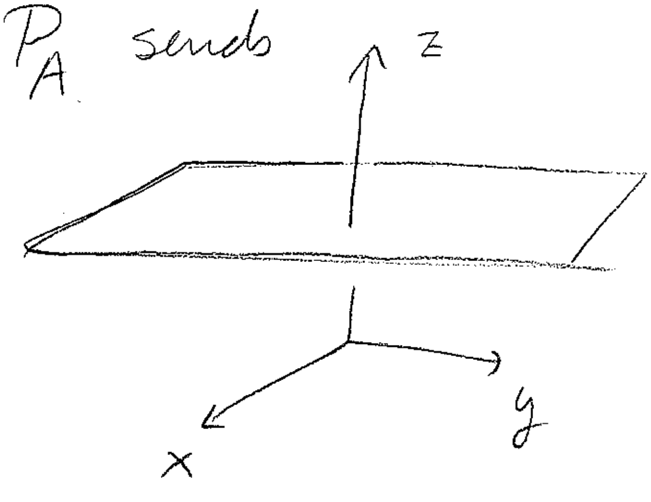
$\mathbb{P}^2_{\mathbb{R}}$ is very symmetric: let $A \in GL_3\mathbb{R}$, i.e. A is a 3×3 matrix with $\det(A) \neq 0$.

Consider the <u>projective transformation</u>:

$$P_A : \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}} \quad \text{via } x \to Ax$$

which makes sense as $\lambda x \mapsto A(\lambda x) = \lambda(Ax)$

<u>Ex</u>: If $A = \begin{pmatrix} B & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 1 \end{pmatrix}$ then P_A sends $\mathbb{R}^2 = \{(x:y:1)\}$ to itself, and acts on it via the linear trans w/ matrix B.



<u>Ex</u>: $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ gives translations.

$$P_A(x:y:1) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+2 \\ y+3 \\ 1 \end{pmatrix} = (x+2 : y+3 : 1)$$

Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad P_A(x:y:z) = (z:x:y)$

Takes $\mathbb{P}^1_\infty = \{(x:y:0)\}$ to

$$\{(0:x:y)\} = \{(0:1:0)\} \cup \underbrace{\{(0:x:1)\}}_{y\text{-axis in } \mathbb{R}^2}$$

[*Moral*: Nothing special about $\mathbb{P}^1_\infty$ — it's just another line.]

Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ $\quad V = \mathbb{V}(\overbrace{xy - z^2}^{f \in \mathbb{R}[x,y,z]})$

Q: What is $P_A(V)$? [Should be another variety...]

Consider
$$\begin{tikzcd} \mathbb{P}^2_{\mathbb{R}} \arrow[r, shift left, "P_{A^{-1}}"'] & \mathbb{P}^2_{\mathbb{R}} \arrow[l, shift left, "P_A"'] \arrow[r, "f"] & \mathbb{R} \\ P_A(V) \arrow[u, phantom, "\cup"] & V \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

and so $P_A(V) = \mathbb{V}(f \circ P_{A^{-1}})$. Use coor

$$\begin{array}{ccc} \mathbb{P}^2_{\mathbb{R}} & \xrightarrow[P_A^{-1}]{} & \mathbb{P}^2_{\mathbb{R}} \\ (u:v:w) & & (x,y,z) \end{array}$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v+w \\ -v+w \\ u \end{pmatrix}$$

and $xy - z^2 = (v+w)(-v+w) - u^2 = -v^2 + w^2 - u^2$

$$V' = p_A(V) = \mathbb{V}_{\mathbb{P}^2_{\mathbb{R}}}(u^2+v^2-w^2) \underset{\substack{\uparrow \\ \text{no pts at } \infty}}{=} \mathbb{V}_{\mathbb{R}^2}(u^2+v^2-1)$$

So in $\mathbb{P}^2_{\mathbb{R}}$, have

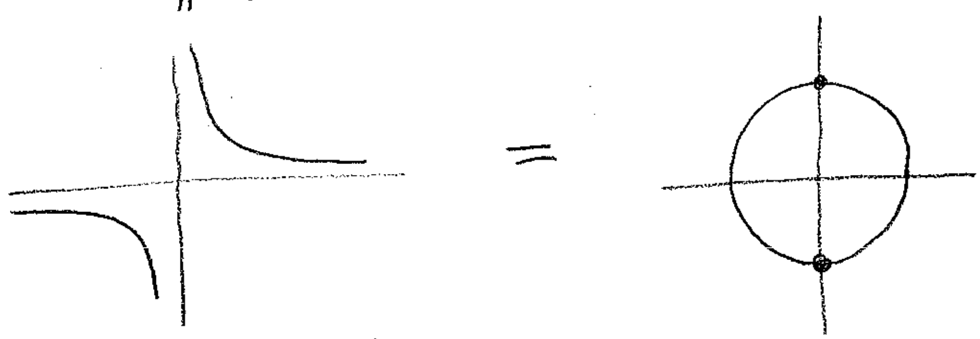


image of $\mathbb{P}^1_\infty$ under p_A

<u>General Conic:</u> In $\mathbb{R}^2$, consider variety of

$$g(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f$$

which comes from

homogenousization

$$V = \mathbb{V}_{\mathbb{P}^2_{\mathbb{R}}}(ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2)$$

Have

$$q(x,y,z) = (x\ y\ z)\overbrace{\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}}^{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which is a quadratic form. There $\exists A \in GL_3\mathbb{R}$ such that $A^T M A = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \varepsilon_3 \end{pmatrix}$ with $\varepsilon_i \in \{-1, 0, 1\}$

[<u>Query</u>: Why? If ness. discuss at length.]

So $V' = P_{A^{-1}}(V)$ is given by

$$q'(u,v,w) = \left(A\begin{pmatrix} u \\ v \\ w \end{pmatrix}\right)^T M \left(A\begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) = (u\ v\ w)\begin{pmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \varepsilon_3 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$= \varepsilon_1 u^2 + \varepsilon_2 v^2 + \varepsilon_3 w^2$$

<u>Thm</u>: Up to a proj. transformation, any conic in $\mathbb{P}^2_{\mathbb{R}}$ is one of

(a) $\mathbb{V}(x^2+y^2-z^2)$ (non-degenerate conic)

(b) $\mathbb{V}(x^2+y^2+z^2) = \emptyset$

(c) $\mathbb{V}(x^2-y^2) = \boxed{\text{HW}}$

(d) $\mathbb{V}(x^2+y^2) = \{(0:0:1)\}$

(e) $\mathbb{V}(x^2) = y$-axis

(f) $\mathbb{V}(0) = \mathbb{P}^2_{\mathbb{R}}$.

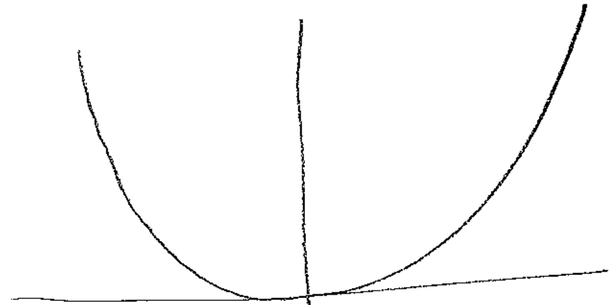
Let C = nondegen conic
L_∞ = line at ∞ = $\{(x:y:0)\}$

Three cases:

$C \cap L = \emptyset \implies$ Ellipse

$C \cap L = 2$ pts $\implies$ Hyperbola

$C \cap L = 1$ pt, tangent intersection $\implies$ parabola.

$\mathbb{V}(x^2-y) =$ 

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$$\mathbb{V}_{\mathbb{P}^2_{\mathbb{R}}}(x^2-yz) \xrightarrow[\begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{pmatrix}]{} \mathbb{V}_{\mathbb{P}^2_{\mathbb{R}}}(v^2-wu) =: V'$$