Lecture the last:

Thm: G a finite group. Then $\exists$ a Galois extension K of $\mathbb{C}(t)$ with $\mathrm{Gal}(K/\mathbb{C}(t)) = G$.

Plan: ① Find an irreducible curve $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ on which G acts by symmetries, and where $V/G = \mathbb{P}^1_{\mathbb{C}} =$

② Then G acts on $\mathbb{C}(V)$ by $\underbrace{\sigma}_{\text{thought of as a sym. of } V} \in G \longmapsto \sigma^*$ with $\sigma^*(f) = f \circ \sigma^{-1}$

③ Set $K = \mathbb{C}(V)$. Then $K_G = \mathbb{C}(V/G) = \mathbb{C}(t)$ and, as always, K/K_G is Galois with group G.

———— ∘ ————

Last time, given G constructed an action on a surface Y via

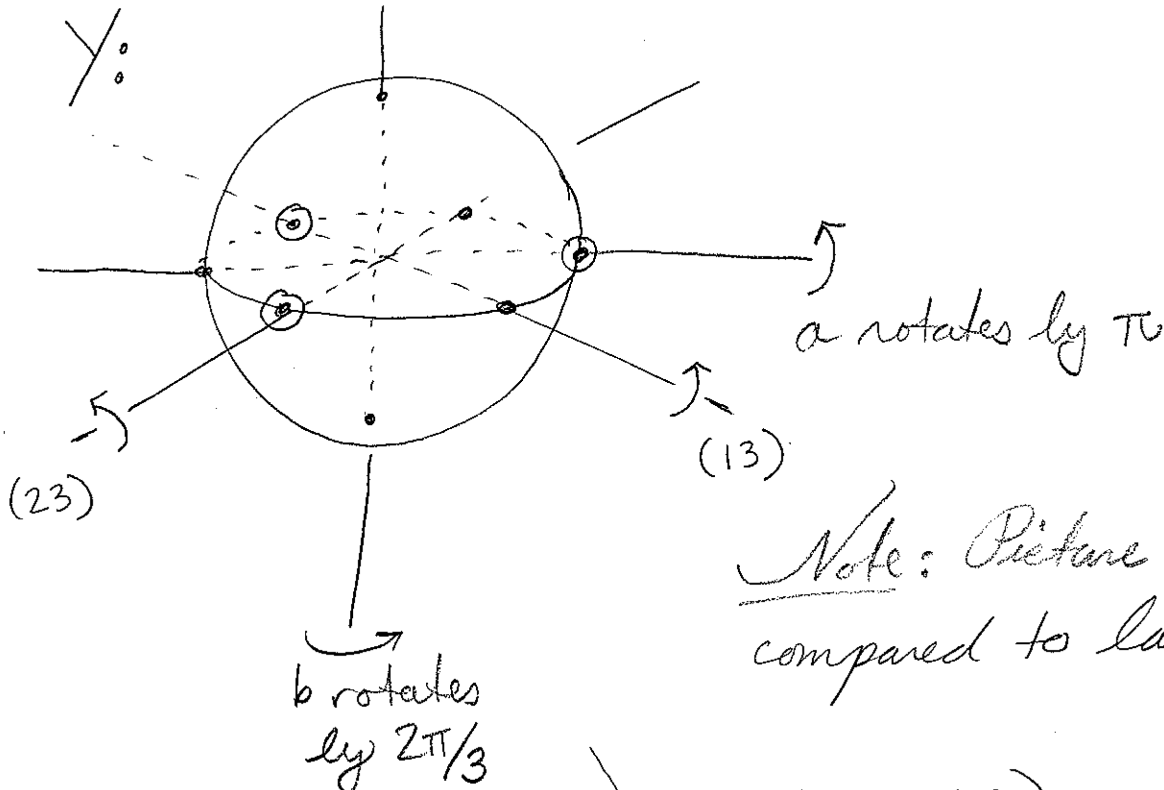
$$G,\ S \text{ gen set} \rightsquigarrow \underset{\substack{\text{Cayley}\\ \text{Graph}}}{\Gamma(G,S)} \overset{\substack{\text{thicken,}\\ \text{add discs}}}{\rightsquigarrow} Y$$

and where $Y/G = X =$ .

[Did this in a specific case, but works in general. Y isn't usually a sphere, though]

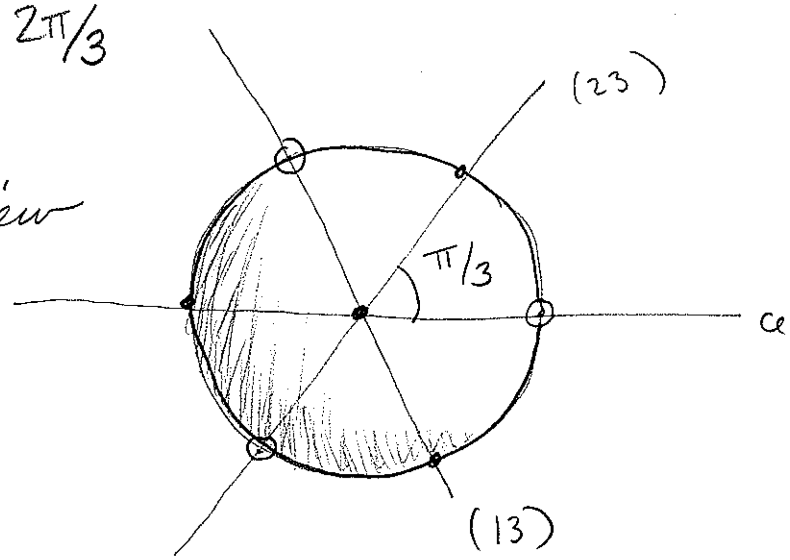
Ex: $G = S_3 = \langle a = (12),\ b = (123) \rangle$

Y:



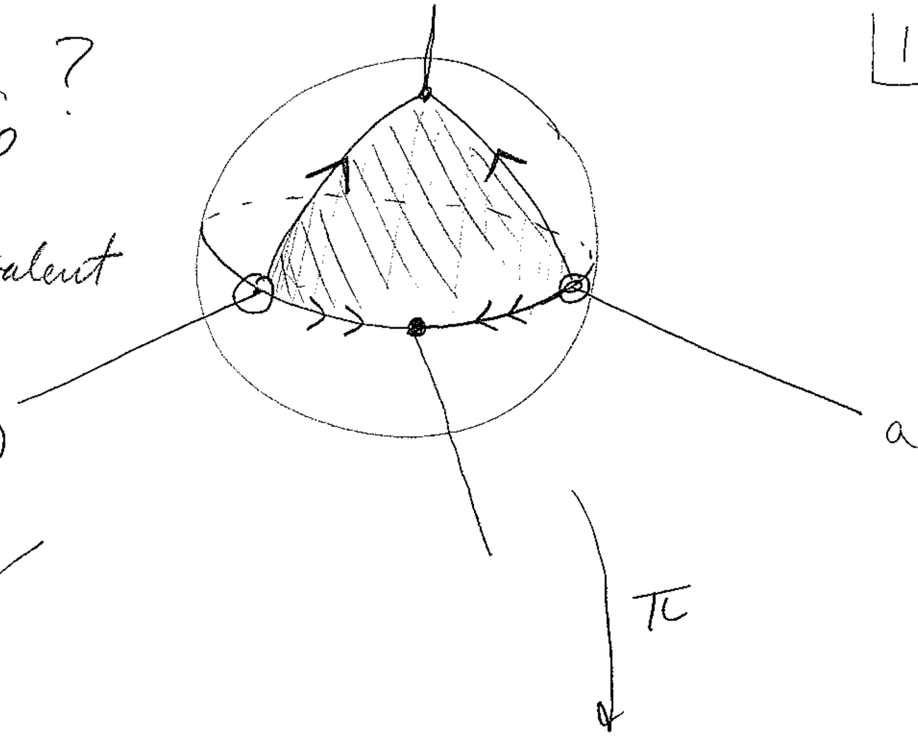
Note: Picture rotated compared to last class.

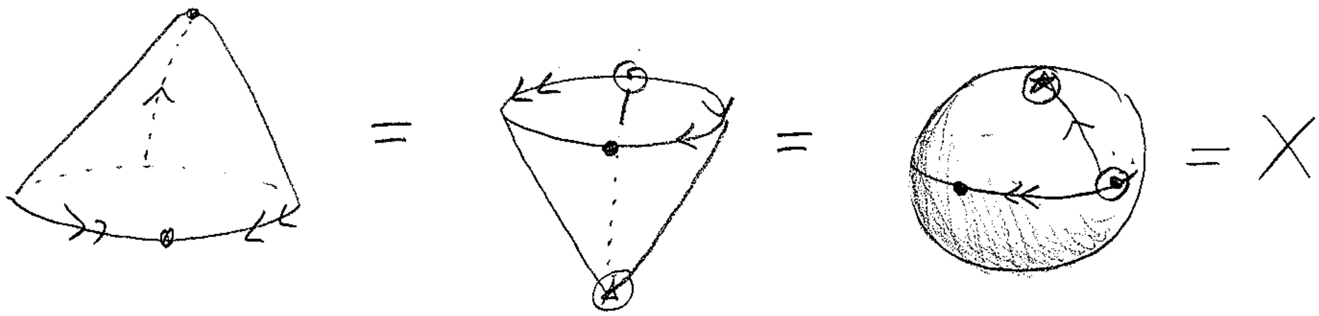
Overhead view

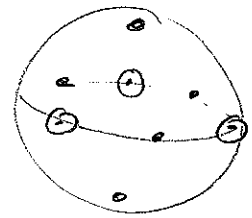


What is $X = Y/G$?

Every pt is equivalent to one in the shaded region.

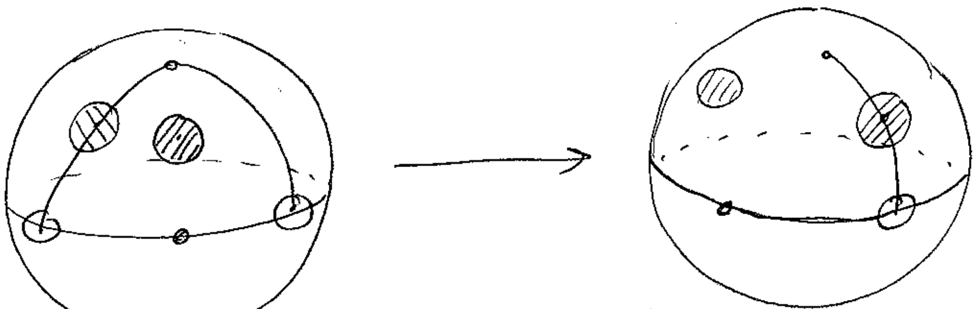




Let B the 8 pts on Y = 

Away from B, the map

$\pi : Y \to X$ is locally 1-1.



Near the poles, map look like

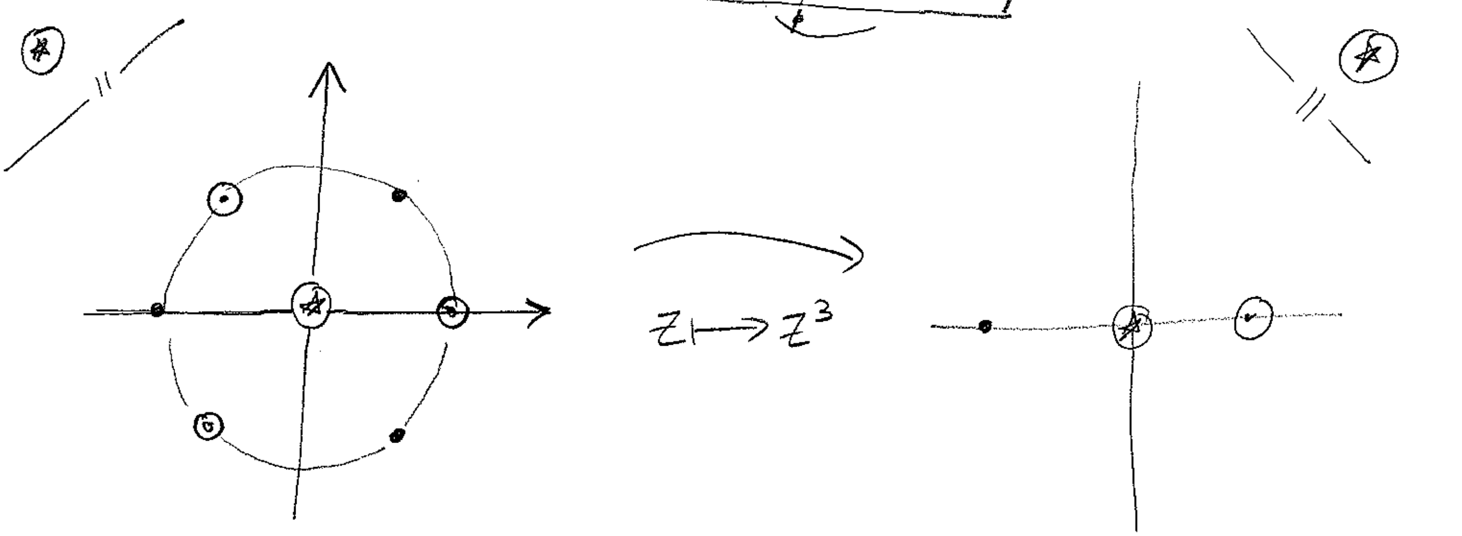
i.e. like $z \longmapsto z^3$ near 0. Near the other pts of B, looks like $z \mapsto z^2$.

<u>So</u>: We've found a surface Y with symmetries G s.t. $X = Y/G$ is 
and so $\pi : Y \to X$ looks locally like a polynomial map (but its just cont).

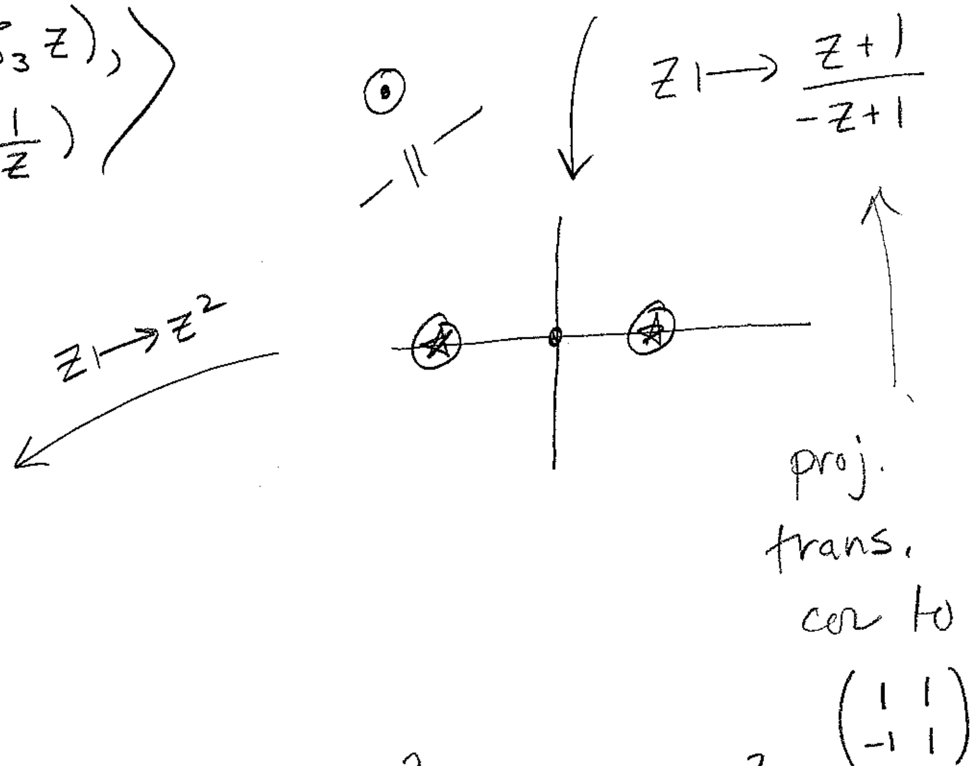
[Aside: π is called a branched covering map.]

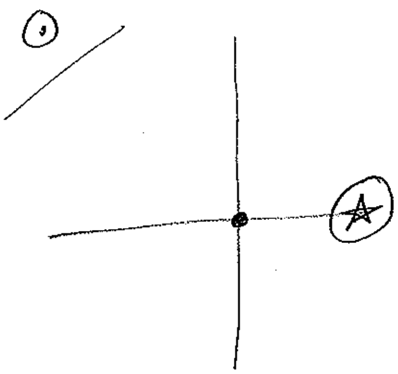
<u>Riemann Existence Thm (Special Case)</u>: There $\exists$ rat'l fn $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ which matches π and so G acts on $\mathbb{P}^1_{\mathbb{C}}$ by projective trans.

Think of $\mathbb{P}^1_{\mathbb{C}}$ as $\mathbb{C} \cup \{\infty\}$ which we can ident with our picture  via stereographic projection



$$G = \left\langle b = (z \mapsto \zeta_3 z),\ a = \left(z \mapsto \tfrac{1}{z}\right) \right\rangle$$





Composit is $h(z) = \left(\frac{z^3+1}{-z^3+1}\right)^2 = \left(\frac{z^3+1}{z^3-1}\right)^2$

Note: Easy to check that if $\sigma \in G$ then $h \circ \sigma = h$, and that h is the quot. map $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}/G$.

Focus on $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ to give

$$\begin{array}{ccc} \mathbb{C}(z) & \xleftarrow{h^*} & \mathbb{C}(t) \\ \left(\frac{z^3+1}{z^3-1}\right)^2 & \longleftarrow & t \end{array}$$

Expanding, get

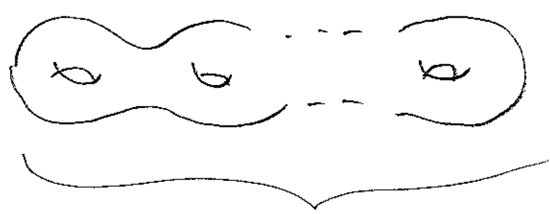
$$(t-1)\, z^6 + 2(t+1) z^3 + t = 0.$$

That is $\mathbb{C}(z)/\mathbb{C}(t) \cong \mathbb{C}(t)[u] \big/ (t-1)u^6 + 2(t+1)u^3 + t$

As we have G acting on $\mathbb{C}(z)$ fixing $\mathbb{C}(t)$ must have $[\mathbb{C}(z) : \mathbb{C}(t)] \geq 6$,

and hence $= 6$. So must be irreducible

Notes: In general, the big field will not be $\cong \mathbb{C}(t)$. In fact, this is only poss when $G = \mathbb{Z}_K, D_K, T, O, I$
tet. oct. icos.

In general, $Y =$ 

g holes

and $|G| \leq 84(g-1)$.

When $g = 3$, the most symmetries is 168. This is realized by the Klein quartic from HW.

Proof uses hyperbolic geometry, also needed for Wiles-Taylor proof of FLT...

The End