Math 418: Takehome 1 due Wednesday, February 18, 2015.

Disclaimer, Terms, and Conditions: You may not discuss the exam with anyone except myself. You may only consult the following:

- The beloved(?) text, Dummit and Foote's *Abstract Algebra*.
- Your class notes and returned HW sets.
- My online class notes and HW solutions.

You can use any result in Chapters 7–9 and Sections 13.1–13.2 of Dummit and Foote, even if I didn’t cover it in class. You can also use the result of any HW problem that was assigned, whether or not you did it.

Office hours: While discussion of these specific problems will be limited to clarifying their statements, I will be happy to answer any broader questions about the course material during my usual office hours (Monday and Tuesday from 2:30–3:30pm), extra office hours (Friday Feb 13, 1:00–2:30pm), or by appointment.

1. Let $F$ be a field. Consider the ring $R = F[[t]]$ of formal power series in $t$, namely things of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 t + a_2 t^2 + \cdots \quad \text{where } a_n \in F.$$  

Here “formal” means the above “sum” is really just an infinite list of elements of $F$; there’s no notion of convergence involved. Elements of $R$ are added term by term, and multiplication is as if they were polynomials. More precisely

$$\sum_{n=0}^{\infty} a_n t^n \times \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) t^n$$

It is clear that $R$ is a commutative ring with unit.

(a) Prove that $\alpha$ in $R$ is a unit if and only if the constant term $a_0 \neq 0$. (Example: The inverse of $1 - t$ is $1 + t + t^2 + t^3 + \cdots$.)

(b) Prove that $R$ is a Euclidean domain with respect to the norm $N(\alpha) = n$ if $a_n$ is the first term of $\alpha$ that is non-zero. (If $F = \mathbb{C}$ and the power series converges near $t = 0$, then this norm is just the order of zero of the corresponding function at 0.)

(c) In the polynomial ring $R[x]$, prove that $x^n - t$ is irreducible.
2. Let \( R = \mathbb{Z}[i] \).

   (a) Prove that \( R/(1 + i) \) is a field of order 2.

   (b) Let \( \pi \in R \) be irreducible. Consider the ideals \( I_n = (\pi^n) \). Prove that \( R/(\pi) \cong I_n/I_{n+1} \) as additive abelian groups. Hint: the isomorphism is multiplication by \( \pi^n \).

   (c) Again for irreducible \( \pi \), prove that \( |R/(\pi^n)| = |R/(\pi)|^n \). Here \( |\cdot| \) denotes the number of elements in a finite set. (This is a key step in proving that for any \( \pi \in R \) that \( |R/(\pi)| = N(\pi) = |\pi|^2 \).

   (d) For \( \pi = 1 + i \), are \( R/(\pi^3) \) and \( \mathbb{Z}/8\mathbb{Z} \) isomorphic as rings?


5. Section 13.2, #20.