Homework #10 Due Wednesday April 26: Rev 2000/4/21

1. Show that Fermat's Last Theorem for exponent 3 implies that the equation $y^2 = x^3 - 432$ has only $(12, \pm 36)$ as rational solutions. Here's an outline of one possible argument: Assume (x, y) is a rational solution different from $(12, \pm 36)$ with $x > 0$.

   (a) Write $y/36 = a/c$, $x/12 = b/c$ with $a \equiv c \equiv 0 \mod 2$.

   (b) Put $r = (a+c)/2, s = (c-a)/2, t = b > 0$.

   (c) Show that $r^3 + s^3 = t^3$, $rst \neq 0$.

2. Show that the converse to the previous exercise is also true. In particular, show that if $x^3 + y^3 = z^3$, $xyz \neq 0$, $x, y, z \in \mathbb{Z}$ then putting $r = 36(x-y)/(x+y), s = 12z/(x+y)$ lead to $r^2 = s^3 - 432$.

3. Using the fact that $x^4 + y^4 = z^2$ has no integral solutions with $xyz \neq 0$ show that $(0, \pm 1)$ are the only rational solutions to $y^2 = x^4 + 1$.

4. Parameterize the rational solutions to the *singular* cubic $y^2 = x^2(x+1)$.

5. For each of the following conics, either find a rational point or prove that there are no rational points:

   (a) $x^2 + y^2 = 6$

   (b) $3x^2 + 5x^2 = 4$

   (c) $3x^2 + 6y^2 = 4$

6. Prove that the curve $y^2 = x^3 - 2$ has infinitely many rational points.

7. The cubic curve $y^2 = x^3 + 17$ has the following five rational points:

   $P_1 = (-2, 3), P_2 = (-1, 4), P_3 = (2, 5), P_4 = (4, 9), P_5 = (8, 23)$

   (a) Show that P_2, P_4, P_5 can each be expressed as $mP_1 + nP_3$ for an appropriate choice of integers m and n.

   (b) Compute the points

   $$P_6 = -P_1 + 2P_3 \quad \text{and} \quad P_7 = 3P_1 - P_3$$

   (c) Notice that the points $P_1, \ldots, P_7$ all have integer coordinates. There is exactly one more rational point on this curve which has integer coordinates and $y > 0$. Find that point.