Homework #2 Due Wed Feb 16

Davenport: 1.15, 1.16, 1.17, 1.19, 2.04, 2.06, 2.14, 2.16, 2.23

N1: If a is a non-zero integer, then for $n > m$ show that $\gcd(a^{2^n} + 1, a^{2^m} + 1)$ is 1 or 2 depending on whether a is odd or even. (Hint: If p is an odd prime and $p \mid a^{2^m} + 1$ then $p \mid a^{2^n} - 1$ for $n > m$).

N2: Use N1 to show there are infinitely many primes.

N3: Show $d(n)$ is odd iff n is a square.

N4: Consider the function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, which is called the Riemann zeta function. It converges for $s > 1$. Euler's identity says:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Provide a "formal proof" of this identity – that is, a proof without worrying about convergence and when you are allowed to swap limits. Letting s take on complex values, $\zeta(s)$ can be extended to the whole complex plane. The Riemann hypothesis states that the only zeros of $\zeta(s)$ lying in the strip $0 \leq \Re(s) \leq 1$ lie on the line $\Re(s) = 1/2$ ($\Re(s)$ denotes the real part of s). The Riemann hypothesis is unproven, and is perhaps the most important open problem in mathematics. A large number of number theory questions could be answered if the Riemann hypothesis was proven.

N5: To get a hint of the connections between the Riemann zeta function and number theory, prove the formal identity:

$$\zeta(s)^2 = \sum_{n=1}^{\infty} d(n)/n^s$$

N6: Show that the equation $3x^2 + 2 = y^2$ has no solutions in integers. (Hint: Show it has no solutions mod n for some n).