Homework #3 Due Wed Feb 23 (Revised Ver)

Davenport: 3.02, 3.04 (through p = 13), 3.07, 3.12.

N1: If $p = 2^n + 1$ is a Fermat prime, show that 3 is a primative root mod p.

N2: The Möbius μ function is another basic number theory function. For $n \in \mathbb{N}$, define $\mu(1) = 1$, $\mu(n) = 0$ if n is not square-free, and $\mu(p_1 p_2 \dots p_k) = (-1)^k$, where the p_i are distinct primes. Suppose $f, g\colon \mathbb{N} \to \mathbb{C}$. The Dirichlet product of f and g, denoted $f \star g$, is defined by $f \star g(n) = \sum_{d_1 d_2 = n} f(d_1) g(d_2)$. You can check that this product is associative. Define the function $\mathbb{I}$ by $\mathbb{I}(1) = 1$ and $\mathbb{I}(n) = 0$ for $n > 1$. Define I by $I(n) = 1$ for all n. Then $f \star I(n) = I \star f(n) = \sum_{d|n} f(d)$.

(a) Prove that if $n > 1$ then $\sum_{d|n} \mu(d) = 0$.

(b) Prove that $I \star \mu = \mu \star I = \mathbb{I}$.

(c) Prove the Möbius inversion theorem: Suppose $f\colon \mathbb{N} \to \mathbb{C}$. Let $F(n) = \sum_{d|n} f(d)$. Then $f(n) = \sum_{d|n} \mu(d) F(n/d)$.

N3: Use the Möbius inversion theorem together with the fact that $\sum_{d|n} \varphi(d) = n$ to rederive the formula for $\varphi(n)$.

N4: Show that the sum of all the primative roots mod p is congruent to $\mu(p-1) \mod p$.

N5: State and prove a theorem which describes what $1^k + 2^k + \dots (p-1)^k$ is congruent to mod p.