Homework #4 Due Wed Feb 30

Davenport: 3.25, 3.26(a-d)

N1: Prove that the equation

$$(x^2 - 13)(x^2 - 17)(x^2 - 221) = 0$$

has solutions mod n for all $n \in \mathbb{N}$, but does not have any integer solutions.

N2: Let p be a prime. Show that the number of solutions to $x^2 - y^2 \equiv a \mod p$ is given by

$$\sum_{y=0}^{p-1} \left(1 + \left(\frac{y^2 + a}{p}\right)\right).$$

N3: Show directly that the number of solutions to $x^2 - y^2 \equiv a \mod p$ is $p - 1$ if p does not divide a and $2p - 1$ if p divides a.

N4: Combine N2 and N3 to show give a formula for

$$\sum_{y=0}^{p-1} \left(\frac{y^2 + a}{p}\right).$$

N5: An integer is a biquadratic residue mod a prime p if it is congruent to a fourth power. Give a necessary and sufficent condition on a prime p such that -4 is a biquadratic residue mod p. Hint: Use the identity $x^4 + 4 = ((x+1)^2 + 1)((x-1)^2 + 1)$.

N6: Use quadratic reciprocity to find the primes for which 7 is a quadratic residue.

N7: Give a direct proof that $(-3/p) = 1$ when $p \equiv 1 \mod 3$. Hint: Show that the cyclic group $U(p)$ has an element ρ of order 3 and look at $(2\rho + 1)^2$.