Homework #5 Due Wed Mar 8

Davenport: 4.03(c), 4.08

N1: Express 310/99 as a continued fraction.

N2: Find the continued fraction expansion of $\sqrt{7}$.

N3: Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Show that at least one out of every 3 convergents p_k/q_k of the continued fraction expansion of x satisfies

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{\sqrt{5} q_k^2}.$$

Therefore, for any irrational x there are infinitely many $p/q \in \mathbb{Q}$ with

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5} q^2}.$$

N4: Show that the result in N3 is the best possible by showing that $\alpha = (1+\sqrt{5})/2$ has the property that for any $c > 0$ there are only finitely many $p/q \in \mathbb{Q}$ with

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{c q^2}.$$

N5: (Note: If this question seems a bit odd, read ahead to problem N6.) Let $\mathrm{GL}_2(\mathbb{Z})$ be the group consisting of those 2 x 2 matrices with integer entries which are invertible. The important thing is that the action of any A in $\mathrm{GL}_2(\mathbb{Z})$ as a linear transformation on $\mathbb{R}^2$ preserves the integer lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$.

1. Prove that a 2 x 2 matrix with entries in $\mathbb{Z}$ is in $\mathrm{GL}_2(\mathbb{Z})$ iff its determinant is ± 1.
2. Let $A \in \mathrm{GL}_2(\mathbb{Z})$. The action of A as a linear transformation on $\mathbb{R}^2$ takes lines to lines. Let $\mathbb{R}\,\mathrm{P}^1$ be the space of lines in $\mathbb{R}^2$ through 0. You can think of $\mathbb{R}\,\mathrm{P}^1$ as being parametrized by the slope of the line in $\mathbb{R} \cup \{\infty\}$. Show that the action of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{R}\,\mathrm{P}^1$ sends the line $y = (1/\alpha)x$ to the line $y = (1/\beta)x$ given by

$$\beta = \frac{a\alpha + b}{c\alpha + d}$$

(This action is a so-called Möbius transformation.)

N6: Two numbers $\alpha, \beta \in \mathbb{R}$ are said to be equivalent if there are integers a, b, c, d with $ad - bc = \pm 1$ such that

$$\beta = \frac{a\alpha + b}{c\alpha + d}$$

You can easily check that this is an equivalence relation.

1. Show that any 2 rational numbers are equivalent.
2. Find a condition on the continued fraction expansions of α and β such that α is equivalent to β iff condition. (Hint: Problem N5)