Homework #7 Due Friday Mar 24

Davenport: 5.08, 6.01, 6.02, 6.05

N1: Show that 23 is the sum of 9 cubes but not the sum of 8 cubes.

N2: For $k \in \mathbb{N}$, let $g(k)$ be the smallest g such that every positive integer is the sum of g k^{th}-powers. Prove that $g(k) \geq 2^k + \lfloor 3^k/2^k \rfloor - 2$. (Hint: Look at $n = 2^k \lfloor 3^k/2^k \rfloor - 1$).

N3: Prove *Thue's Lemma:* Let p be an odd prime, a an integer not divisible by p, then there are integers x and y such that $ax \equiv y \mod p$, with $0 < |x| < \sqrt{p}$, $0 < |y| < \sqrt{p}$.

N4: Use Thue's Lemma to give another proof that a prime congruent to 1 mod 4 is the sum of two squares. (Apply Thue to an a with $a^2 \equiv -1 \mod p$.)

N5: Let $f(x, y) = ax^2 + bxy + cy^2$ be a primative quadratic form, and Δ its discriminant. Prove there is a one-to-one correspondence between automorphisms of f and solutions to the equation

$$x^2 - \Delta y^2 = 4.$$

(On the next HW, you'll use this result to show that the above equation always has a solution.).

N6: Prove that the automorphism group of a *definite* quadratic form is finite (that is, there are only finitely many automorphisms). Some of the most beautiful simple groups arise as automorphism groups of quadratic forms in higher dimensions. What about the automorphism groups of indefinite forms?