Homework #7 Due Wednesday April 12: Revision 2

Davenport: 6.02, 6.05, 6.08

N1: Prove that an indefinite quadratic form f represents 0 (that is, there is an $(x, y) \neq (0, 0)$ with $f(x, y) = 0$) if and only if the discriminant of f is a square.

N2: Let $f = (a, b, c)$ be a binary quadratic form. Let ω_f be its principle root. Prove that if $A \in \mathrm{SL}_2\mathbb{Z}$ then

$$\omega_f = \frac{\alpha\omega_{f\circ A} + \beta}{\delta\omega_{f\circ A} + \gamma} \quad \text{where} \quad A = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}.$$

N3: Let $f(x, y) = ax^2 + bxy + cy^2$ be a primitive quadratic form, and Δ its discriminant. Prove there is a one-to-one correspondence between automorphisms of f and solutions (in $\mathbb{Z}$) to the equation

$$x^2 - \Delta y^2 = 4.$$

N4: Prove that the automorphism group of a *definite* quadratic form is finite (that is, there are only finitely many automorphisms). Some of the most beautiful simple groups arise as automorphism groups of quadratic forms in higher dimensions. What about the automorphism groups of indefinite forms?

N5: Suppose $\Delta > 0$ is the discriminant of an indefinate quadratic form. Use N3 to prove that the equation

$$x^2 - \Delta y^2 = 4.$$

has a solution in $\mathbb{N}$. Use this to give another proof that Pell's equation

$$x^2 - \Delta y^2 = 1.$$

has a solution in $\mathbb{N}$.

N6: Let $\Delta > 0$ be the discriminant of an indefinate quadratic form. Prove that the equation

$$x^2 - \Delta y^2 = -4$$

has solutions if and only if there are reduced forms $(1, b, c)$ and $(-1, b, -c)$ which lie in the same cycle. What does this result say about solutions to the equation

$$x^2 - \Delta y^2 = -1?$$

N7: Find the number of forms of discriminant 21.