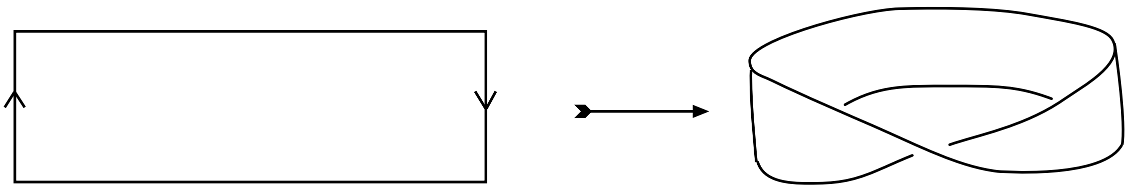
Math 131 Homework #11; Due Tuesday April 22

Munkres: 9.59 1, 3, 4.

N1: Let X be a Möbius band, that is, a square with a pair of opposite sides identified by a twist as shown in the figure:



(Formally, you can define X as a quotient space of $I \times I$; see Munrkes page 450 and §2.22).

(a) Show that $\pi_1(X) = \mathbb{Z}$.

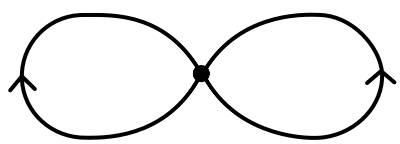
(b) Exhibit a cover $p\colon Y \to X$ where Y is simply connected.

(c) Show that there does not exist a retract of X to its boundary circle C.

N2: Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ must there exist a $(x,y) \in S^1 \times S^1$ such that $f(x,y) = f(-x,-y)$?

N3: Let A_1, A_2, A_3 be closed subsets of S^2 which cover S^2. Then at least one of these sets must contain a pair of antipodal points $\{-x, x\}$. Hint: Look at the functions d_i which measure the distance to A_i.

N4: Let X be the figure-8:

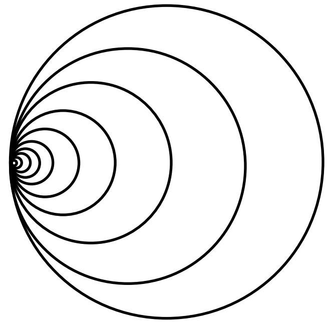


Let x_0 be the marked point at the center. Let $a = [f]$ and $b = [g]$ where where f is a path going around the right loop once and g a path going around the left one once (in the directions shown). We know from class that $\pi_1(X, x_0)$ is generated by a and b.

(a) Consider the element $g = aba^{-1}b^{-1} \in \pi_1(X, x_0)$. Give a covering map $p\colon Y \to X$ such that $p^{-1}(x_o)$ is finite and so that g has a lift to a path with distinct endpoints.

(b) Let F be the free group on a and b, and let $\phi\colon F \to \pi_1(X, x_o)$ be the induced map. Give an alternate proof that ϕ is an isomorphism as follows. Let $w \neq 1$ be a word in F. Show that $\phi(w) \neq 1$ in $\pi_1(X, x_0)$ by finding a *finite* cover $p\colon Y \to X$ such that $\phi(w)$ lifts to a path with distinct endpoints.

N5: Consider the subspace X of $\mathbb{R}^2$ which is the union of circles C_n of radius $1/n$ and center $(1/n, 0)$ for $n \in \mathbb{Z}_+$. This space is called the *Hawaiian Earring*.



(a) Despite the fact that X is just a compact metric space, show that $\pi_1(X)$ is uncountable as follows: Let $r_n\colon X \to C_n$ be a retraction which collapse all C_i's to the origin except C_n. Then r_n gives a homomorphism $\rho_n\colon \pi_1(X) \to \pi(C_n) \cong \mathbb{Z}$. Together, the ρ_n induce a homomorphism $\rho\colon \pi_1(X) \to \prod_{n\in\mathbb{Z}_+} \mathbb{Z}$. (Recall that $\prod_{\mathbb{Z}_+} \mathbb{Z}$ is the direct product, not the direct sum; that is if $g = (g_n) \in \prod_{n\in\mathbb{Z}_+} \mathbb{Z}$ then infinitely many g_n are allowed to be non-zero.) Show that ρ is surjective and thus that $\pi_1(X)$ is uncountable.

(b) Is ρ injective, i.e. is $\pi_1(X) \cong \prod_{n\in\mathbb{Z}_+} \mathbb{Z}$?

Note: Shelah showed in 1988 that a path-connected, locally path-connected compact metric space X, $\pi_1(X)$ is either finitely generated or uncountable. Shelah is primarily a logician who has written more than 800 papers!