Annoying trailers:

SnapPy http://snappy.computop.org

What is SnapPy?

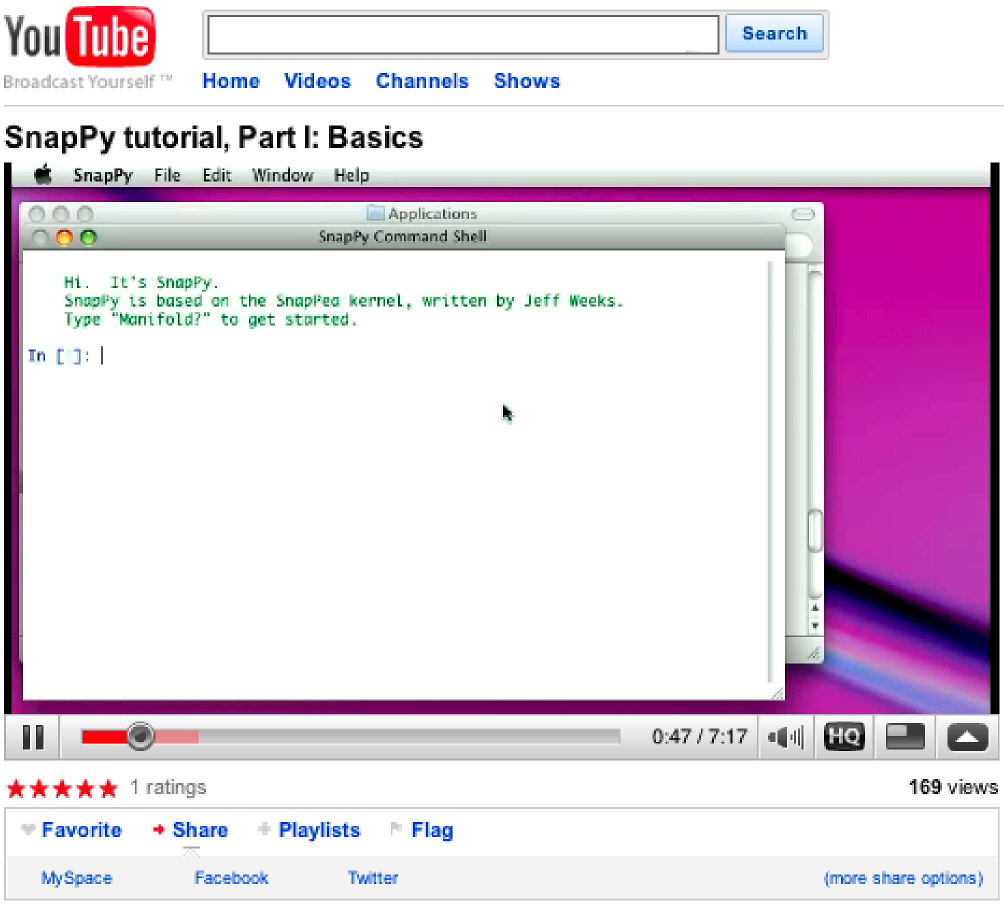
SnapPy is a user interface to the SnapPea kernel which runs on Mac OS X, Linux, and Windows. SnapPy combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the Python programming language. You can see it in action, learn how to install it, and read the tutorial.

Contents

Credits

Written by Marc Culler and Nathan Dunfield. Uses the SnapPea kernel written by Jeff Weeks. Released under the terms of the GNU General Public License.

http://www.youtube.com/user/NathanDunfield/



People who heard this talk also viewed:

- J. Aaber and N. Dunfield
  Closed surface bundles of least volume
  arXiv:1002.3423

Hyperbolically twisted Alexander polynomials of knots

Nathan M. Dunfield
University of Illinois

Stefan Friedl
Nicholas Jackson
Warwick

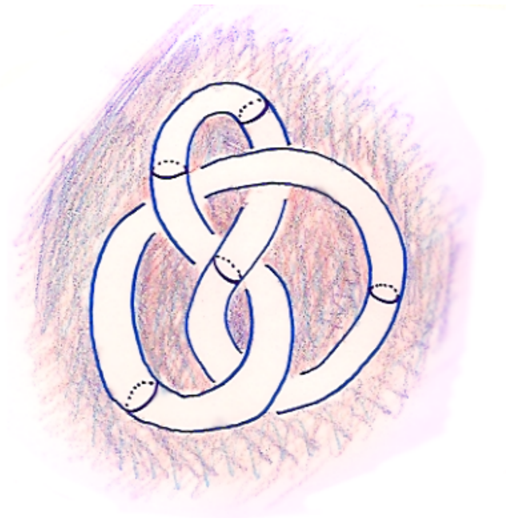
Jacofest, June 4, 2010

This talk available at http://dunfield.info/

Math blog: http://ldtopology.wordpress.com/

Setup:

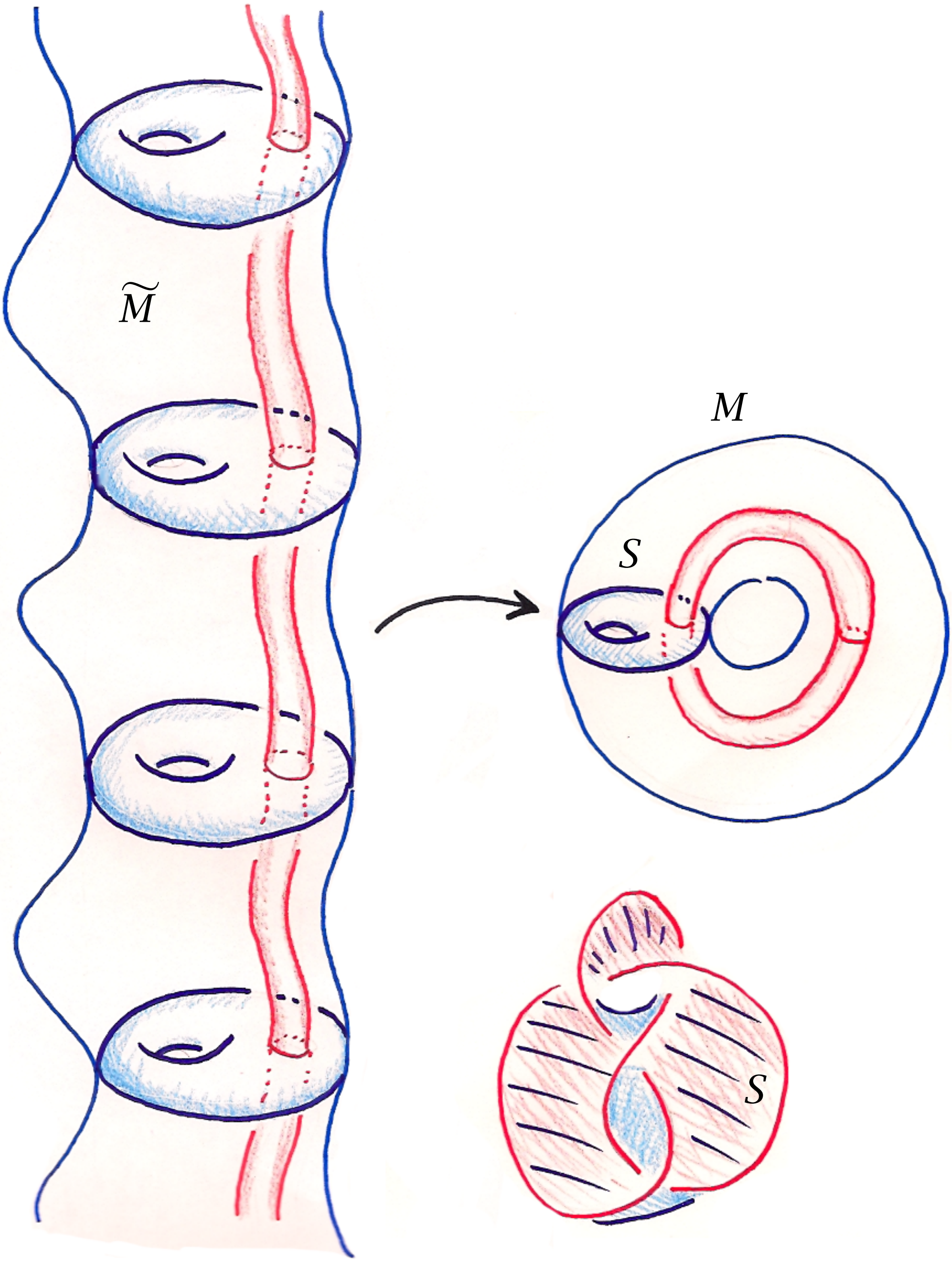
- Knot: $K = S^1 \hookrightarrow S^3$
- Exterior: $M = S^3 - \mathring{N}(K)$



A basic and fundamental invariant of K its *Alexander polynomial* (1923):

$$\Delta_K(t) = \Delta_M(t) \in \mathbb{Z}[t, t^{-1}]$$

Universal cyclic cover: corresponds to the kernel of the unique epimorphism $\pi_1(M) \to \mathbb{Z}$.



$A_M = H_1(\widetilde{M};\mathbb{Q})$ is a module over $\Lambda = \mathbb{Q}[t^{\pm 1}]$, where $\langle t \rangle$ is the covering group.

As Λ is a PID,

$$A_M = \prod_{k=0}^{n} \Lambda \Big/ (p_k(t))$$

Define

$$\Delta_M(t) = \prod_{k=0}^{n} p_k(t) \in \mathbb{Q}[t, t^{-1}]$$

Figure-8 knot:

$$\Delta_M = t - 3 + t^{-1}$$

Genus:

$$
\begin{aligned}
g &= \min\left(\text{genus of } S \text{ with } \partial S = K\right) \\
&= \min\left(\text{genus of } S \text{ gen. } H_2(M, \partial M; \mathbb{Z})\right)
\end{aligned}
$$

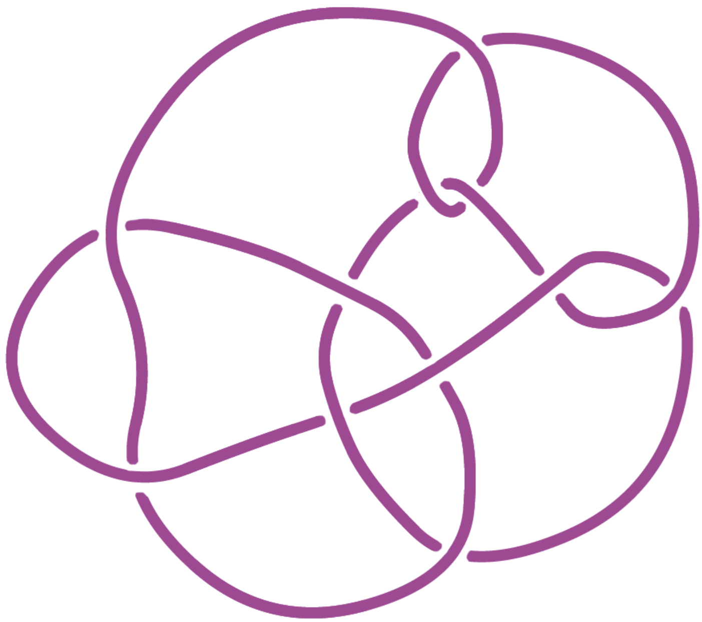
Fundamental fact:

$$2g \geq \deg(\Delta_M)$$

Proof: Note $\deg(\Delta_M) = \dim_{\mathbb{Q}}(A_M)$. As A_M is generated by $H_1(S; \mathbb{Q}) \cong \mathbb{Q}^{2g}$, the inequality follows.

$\Delta(t)$ determines g for all alternating knots and all fibered knots.

Kinoshita-Terasaka knot: $\Delta(t) = 1$ but $g = 2$.



Focus: Improve Δ_M by looking at $H_1(\widetilde{M}; V)$ for some system V of local coefficients.

Assumption: M is hyperbolic, i.e.

$$\mathring{M} = \mathbb{H}^3 \big/ \Gamma \quad \text{for a lattice } \Gamma \leq \mathrm{Isom}^+ \mathbb{H}^3$$

Thus have a faithful representation

$$\alpha\colon\ \pi_1(M) \to \mathrm{SL}_2\mathbb{C} \leq \mathrm{Aut}(V) \quad \text{where } V = \mathbb{C}^2.$$

Hyperbolic Alexander polynomial:

$$\tau_M(t) \in \mathbb{C}[t^{\pm 1}] \quad \text{coming from } H_1(\widetilde{M}; V_\alpha).$$

Examples:

- Figure-8: $\tau_M = t - 4 + t^{-1}$
- Kinoshita-Terasaka:

$$\begin{aligned}\tau_M \approx &(4.417926 + 0.376029i)(t^3 + t^{-3})\\ &- (22.941644 + 4.845091i)(t^2 + t^{-2})\\ &+ (61.964430 + 24.097441i)(t + t^{-1})\\ &- (-82.695420 + 43.485388i)\end{aligned}$$

Really best to define $\tau_M(t)$ as torsion, a la Reidemeister/Milnor/Turaev.

Basic Properties:

- Can be normalized so $\tau_M(t) = \tau_M(t^{-1})$.
- Then τ_M is an actual element of $\mathbb{C}[t^{\pm 1}]$, in fact of $\mathbb{Q}\big(\operatorname{tr}(\Gamma)\big)[t^{\pm 1}]$.
- $\tau_{\overline{M}} = \overline{\tau_M(t)}$
- M amphichiral $\Rightarrow \tau_M(t) \in \mathbb{R}[t^{\pm 1}]$.
- $\tau_M(\zeta) \neq 0$ for any root of unity ζ.
- Genus bound:

$$4g - 2 \geq \deg \tau_M(t)$$

For the KT knot, $g = 2$ and $\deg \tau_M(t) = 3$ so this is sharp, unlike with Δ_M.

Knots by the numbers:

313,231	number of prime knots with at most 15 crossings. [HTW 98]
8,834	number where $2g > \deg(\Delta_M)$.
22	number which are non-hyperbolic.

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22	number which are non-hyperbolic.
0	number where $4g - 2 > \deg(\tau_M)$.

Conj. *τ_M determines the genus for any hyperbolic knot in S^3.*

Computing τ_M: Approximate $\pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ to 250 digits by solving the gluing equations associated to some ideal triangulation of M to high precision.

Many properties of M^3 are algorithmically computable, including

[Haken 1961] Whether a knot K in S^3 is unknotted. More generally, can find the genus of K.

[Jaco-Oertel 1984] Whether M contains an incompressible surface.

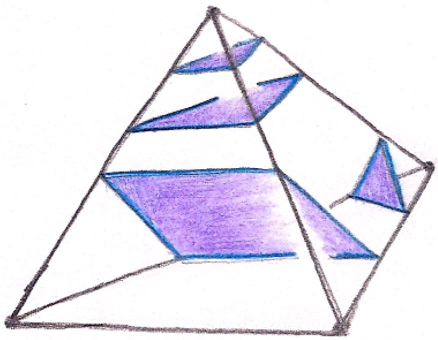
[Rubinstein-Thompson 1995] Whether M is S^3.

[Haken-Hemion-Matveev] Whether two Haken 3-manifolds are homeomorphic.

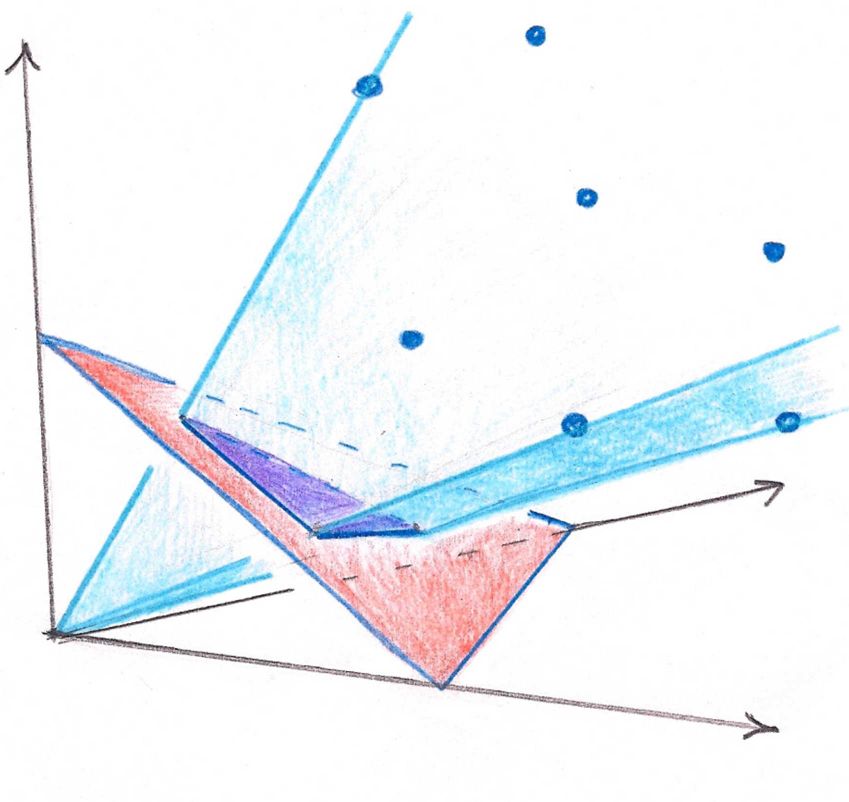
All of these plus Perelman, Thurston, Casson-Manning, Epstein et. al., Hodgson-Weeks, and others give:

Thm. *There is an algorithm to determine if two compact 3-manifolds are homeomorphic.*

Normal surfaces meet each tetrahedra in a triangulation $\mathcal{T}$ of M in a standard way:



and correspond to certain lattice points in a finite polyhedral cone in $\mathbb{R}^{7t}$ where $t = \#\mathcal{T}$:



Meta Thm. *In an interesting class of surfaces, there is one which is normal. Moreover, one lies on a vertex ray of the cone.*

E.g. The class of minimal genus surfaces whose boundary is a given knot.

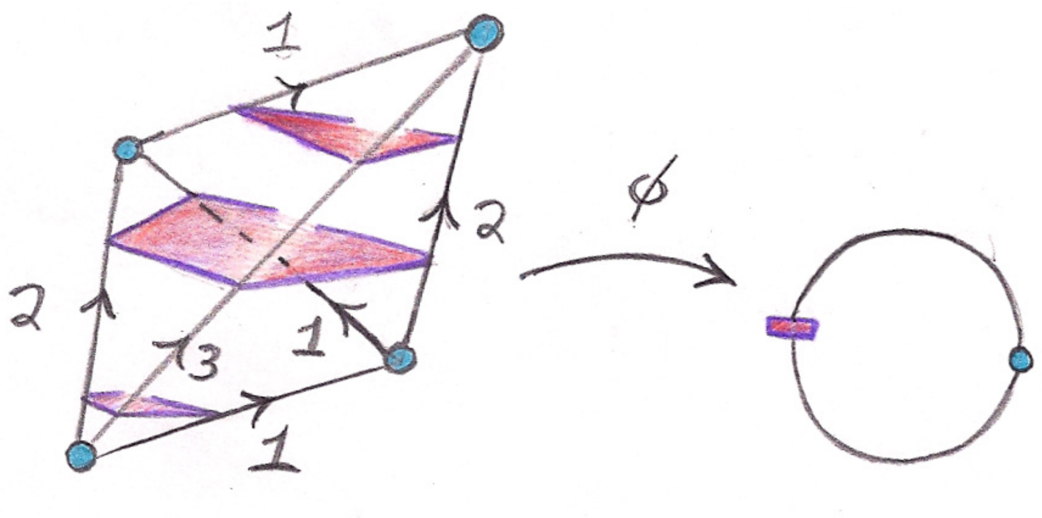
Problem: There can be exponentially many vertex rays, typically $\approx O(1.6^t)$ [Burton 2009]. In practice, limited to $t < 40$.

[Agol-Hass-Thurston 2002] Whether the genus of a knot $K \subset M^3$ is $\leq g$ is NP-complete.

[Agol 2002] When $M = S^3$ the previous question is in co-NP.

Practical Trick: Finding the simplest surface representing some $\phi \in H^1(M;\mathbb{Z}) \cong H_2(M,\partial M;\mathbb{Z})$.

Take a triangulation with only one vertex (cf. Jaco-Rubinstein, Casson). Then ϕ comes from a unique 1-cocycle, which realizes ϕ as a piecewise affine map $M \to S^1$.



Power of randomization: Trying several different $\mathcal{T}$ usually yields the minimal genus surface.

Basic Fact: If M fibers over the circle then τ_M is monic, i.e. lead coefficient ± 1.

Current focus: For 15 crossing knots, does τ_M determine whether M fibers?

By Gabai can reduce to the case of *closed* manifolds.

Practical Trick: Proving that $N = M \setminus \Sigma$ is $\Sigma \times I$.

Start with a presentation for $\pi_1(N)$ coming from a triangulation, and then simplify that it using Tietze transformations. With luck (i.e. randomization), one gets a one-relator presentation of a surface group. This gives $N \cong \Sigma \times I$ by [Stallings 1960].

[Dunfield-Ramakrishnan 2008] Used this when $|\mathcal{T}| > 130$.

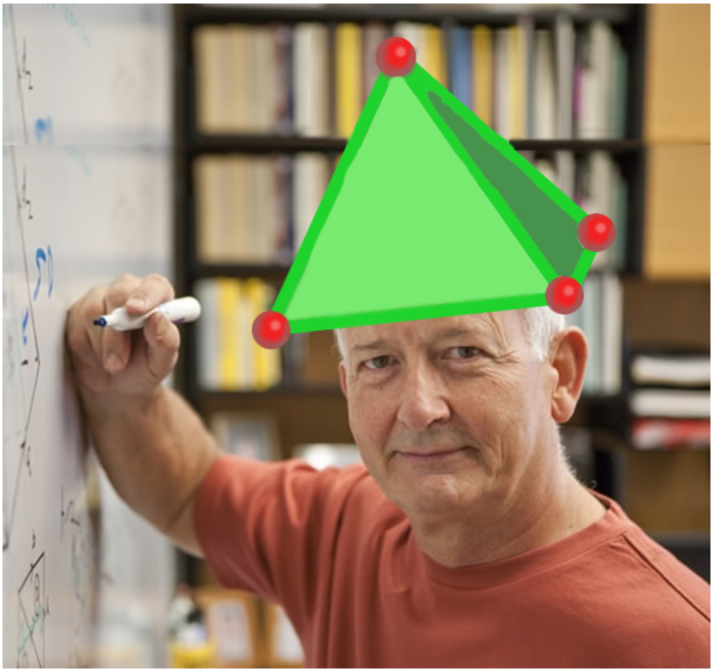
General approach uses Jaco-Rubinstein "crushing". Compare [Burton-Rubinstein-Tillmann 2009].

Future work: Considering τ_M as a function on the character variety.

Generic goals:

- Explain why genus bounds of τ_M are as good as those of Δ_M.
- Use ideal points associated to Seifert surfaces to show nonfibered implies τ_M is non-monic.
- Genus info?

Happy Birthday



Bus!