HIGHLY CONNECTED MONOCHROMATIC SUBGRAPHS: ADDENDUM

HENRY LIU, ROBERT MORRIS AND NOAH PRINCE

Department of Mathematical Sciences,
The University of Memphis, Memphis, TN 38152

Abstract. We consider the following conjecture of Bollobás and Gyárfás: Let $n \geq 4n - 3$. Then, in any 2-colouring of $E(K_n)$, one can find a k-connected, monochromatic subgraph on at least $n-2k+2$ vertices. We shall prove the case of this conjecture when $k = 3$.

1. Introduction

The following observation of Erdős is well known and easy to prove: For any graph G, either G or $\overline{G}$ is connected. Bollobás and Gyárfás [2] considered the following extension of this observation: In any 2-colouring of $E(K_n)$, on how many vertices are we guaranteed to be able to find a monochromatic, k-connected subgraph?

A graph G is *k-connected* if $|V(G)| \geq k+1$, and whenever $C \subset V(G)$ with $|C| \leq k - 1$, we have the graph $G - C$ is connected.

Bollobás and Gyárfás made the following conjecture.

Conjecture 1 (Bollobás and Gyárfás, 2003). *For $n \geq 4k - 3$, whenever we 2-colour the edges of K_n, we can find a k-connected, monochromatic subgraph on at least $n - 2k + 2$ vertices.*

The case $k = 1$ of Conjecture 1 is Erdős' observation, while the case $k = 2$ is also easy to prove [2]. In this note, we shall prove the case of Conjecture 1 when $k = 3$.

For the rest of this note, we shall assume that in any 2-colouring, the colours used are blue and red.

2. Constructions

Bollobás and Gyárfás considered the following 2-colouring of $E(K_n)$, which led the the suggestion of Conjecture 1.

Construction 1. Let $n \geq 4k - 3$. Partition $V(K_n)$ into $V(K_n) = \bigcup_{i=1}^{4} V_i \cup W$, where $|V_i| = k - 1$ for $1 \leq i \leq 4$ and $|W| = n - 4(k - 1)$. Colour all edges in $E(V_1, V_2)$, $E(V_3, V_4)$, $E(V_2, V_4)$, $E(V_2, W)$, $E(V_4, W)$ blue, all other edges between the classes red, and all edges within the classes arbitrarily. See Figure 1.

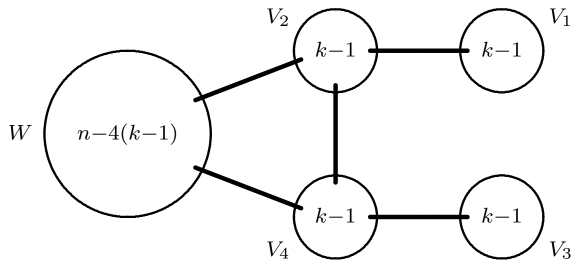


FIGURE 1. The blue-red colouring of $E(K_n)$ described by Construction 1.

Figure 1 shows the edges which are necessarily blue, where the thick lines here, and in subsequent diagrams, represent all the possible edges between the relevant classes. All other edges between the classes are red, and all edges within the classes are arbitrarily coloured.

It is an easy exercise to show that the largest (in order) k-connected, monochromatic subgraph in any blue-red colouring of the form described in Construction 1 has $n - 2k + 2$ vertices.

Also, Construction 2 below shows that we cannot get a k-connected monochromatic subgraph *at all* if $n \leq 4k - 4$. In fact, in Construction 2, we shall describe a colouring using r colours which shows that we cannot get a k-connected monochromatic subgraph at all if $n \leq 2r(k-1)$.

Construction 2. Let $r \geq 2$. It is well-known that K_{2r} has a decomposition into r edge-disjoint Hamilton paths $P^{(1)}, \dots, P^{(r)}$ (for example, see [1], Ch. I, Theorem 11). Figure 2 below shows the case when $r = 3$.

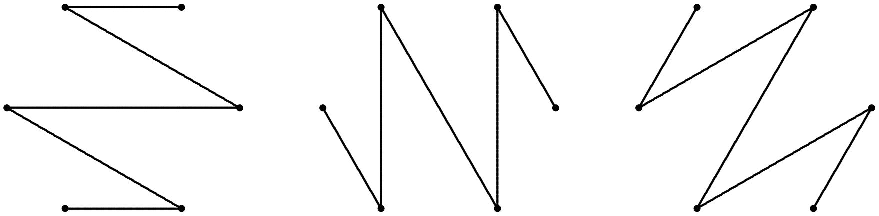
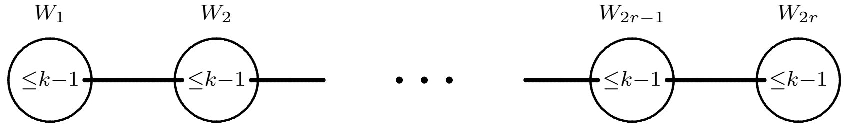


FIGURE 2. A decomposition of K_{2r} into Hamilton paths.

Let $V(K_{2r}) = \{v_1, \dots, v_{2r}\}$. Now, partition $V(K_n)$ into $V(K_n) = \bigcup_{i=1}^{2r} V_i$, where $|V_i| \leq k-1$ for $1 \leq i \leq 2r$. Colour all edges in

$E(V_i, V_j)$, with colour m iff $v_iv_j \in E(P^{(m)})$. Also, colour all edges within V_i with colour m iff v_i is an end-vertex of $P^{(m)}$. Figure 3 below shows the form that the subgraph which is induced by one of the colour classes must take: it is a 'blown-up P_{2r-1}', with the end-vertices replaced by cliques and the other vertices replaced by independent sets.



along with all edges within W_1 and W_{2r}

FIGURE 3. The subgraph induced by one of the colours in Construction 2.

In Figure 3, $\{W_1, \ldots, W_{2r}\} = \{V_1, \ldots, V_{2r}\}$, and $|W_i| \leq k-1$ for $1 \leq i \leq 2r$.

It is another easy exercise to show that in any blue-red colouring of the form described in Construction 2, we do not have a k-connected, monochromatic subgraph *at all*.

Constructions 1 and 2 together show that Conjecture 1 is really the strongest possible, in the sense that the bound of $n - 2k + 2$ for the largest k-connected monochromatic subgraph cannot be larger, and the bound of $4k - 3$ for n cannot be smaller.

3. Proof of Conjecture 1 when $k = 3$

In this section, we shall prove the case of Conjecture 1 when $k = 3$.

Throughout this section, we let $V = V(K_n)$, and let $B \subset K_n$ denote the subgraph with vertex set V and with the blue edges, and define R similarly. Also, we write $K_{i,j}^{(B)}(t)$ for a complete bipartite graph with blue edges and with class sizes i and j, where $1 \leq i \leq j$ and $i + j = t$, and similarly for R in place of B. For example, $K_{i,j}^{(B)}(5)$ can be a blue $K_{1,4}$ or a blue $K_{2,3}$. We will drop the t if i and j are known. For example, $K_{2,4}^{(R)}$ is a red $K_{2,4}$.

We shall first make the following well-known observation.

Observation 1. *If R is not k-connected, then B must contain a $K_{i,j}^{(B)}(n-k+1)$. A similar statement holds when we switch B and R.* □

We are now ready to prove the theorem.

Theorem 2. *Let $n \geq 9$. Then, in any 2-colouring of $E(K_n)$, we can find a 3-connected, monochromatic subgraph on at least $n-4$ vertices. The bound $n-4$ cannot be replaced by a larger function.*

Proof. Take an arbitrary blue-red colouring of $E(K_n)$, where $n \geq 9$. By Constructions 1 and 2, it suffices to show that we can find a monochromatic, k-connected subgraph on at least $n-4$ vertices.

If R is not 3-connected, then by Observation 1, we must have a $K_{i,j}^{(B)}(n-2)$. This graph is 3-connected when $i \geq 3$. When $i = 2$, if we do not have a red 3-connected graph on the j-class, then by Observation 1, we have a $K_{i',j'}^{(B)}(n-6)$ within the j-class. With the two vertices of the i-class, we have a blue 3-connected subgraph on $n-4$ vertices.

Hence, $i = 1$, and so there is a vertex $x \in V$ with $d_B(x) \geq n-3$. Similarly, there is a vertex $y \in V$ with $d_R(y) \geq n-3$. Without loss of generality, let $d_B(x) = d_R(y) = n-3$ and $xy \in E(R)$. Let $A = \{u \in V : xu \in E(B)\}$ (so, $|A| = n-3$). We now divide our argument into three cases.

Case 1. There exists $a \in A$ with $d_{B[A]}(a) = n-4$.

We are done in this case by the same argument as earlier, since we have a $K_{2,n-4}^{(B)}$.

Case 2. There exists $a \in A$ with $d_{B[A]}(a) = n-5$.

If $R[A \setminus \{a\}]$ is 3-connected, then we are done, so assume otherwise. Then by Observation 1, we have b, $c \in A \setminus \{a\}$ such that we have a $K_{i,j}^{(B)}(n-6)$ on $A \setminus \{a,b,c\}$. If $i \geq 2$ (so that $n \geq 10$), then $B[U]$ is a blue 3-connected subgraph on $n-4$ vertices, where U consists of a, x, and the vertices of the $K_{i,j}^{(B)}(n-6)$. Indeed, if we have a cut-set of $B[U]$ with two vertices, then clearly x must be in the cut-set. It is then easy to check that $B[U]-x$ is 2-connected. Hence, we have $i = 1$ (so that we now have a $K_{1,n-7}^{(B)}$ on $A \setminus \{a,b,c\}$). Moreover, it is easy to see that there exists a unique vertex d in the $(n-7)$-class of the $K_{1,n-7}^{(B)}$ such that $ad \in E(R)$. Let $\{e\}$ be the 1-class. We have a blue $K_3 + E_{n-8}$ on $\{a,e,x\} \cup C$, where $C = A \setminus \{a,b,c,d,e\}$. We make the following observations.

Observation 2. *If $n \geq 9$, then any graph formed by sending three edges from a vertex to anywhere in $K_3 + E_{n-8}$ is 3-connected.* □

Observation 3. *If $n \geq 9$, then any graph formed by sending two edges each from two adjacent vertices u and v to anywhere in $K_3 + E_{n-8}$ is 3-connected, provided that $\Gamma(u) \neq \Gamma(v)$ in $K_3 + E_{n-8}$.* □

Observation 4. *If $n \geq 10$, then any graph formed by sending three edges from a vertex to anywhere in $P_2 + E_{n-8}$ is 3-connected.* □

Since $ba, bx, ca, cx, de, dx \in E(B)$, by Observation 2, we must have b, c, d all sending red edges to C, and that be, $ce \in E(R)$. By Observation 3, we need bd, $cd \in E(R)$. We have a red $P_2 + E_{n-8}$ on $\{b, c, d\} \cup C$. Since y sends at least $n-7$ red edges to this $P_2 + E_{n-8}$, by Observation 4, we will have a red 3-connected subgraph on $(A \setminus \{a, e\}) \cup \{y\}$ if $n \geq 10$. If $n = 9$, let $C = \{f\}$, and let z be the remaining vertex of V. If z sends at least three blue edges to $\{b, c, d, e, f\}$, then we will have a blue 3-connected subgraph on at least six vertices. Indeed, if z sends three blue edges to any three of $\{b, c, e, f\}$, then this 3-set, along with $\{a, x, z\}$, form a blue $K_{3,3}$. So, if $zd \in E(B)$, then Figure 4 shows the possible blue 3-connected subgraphs that can arise, up to isomorphism.

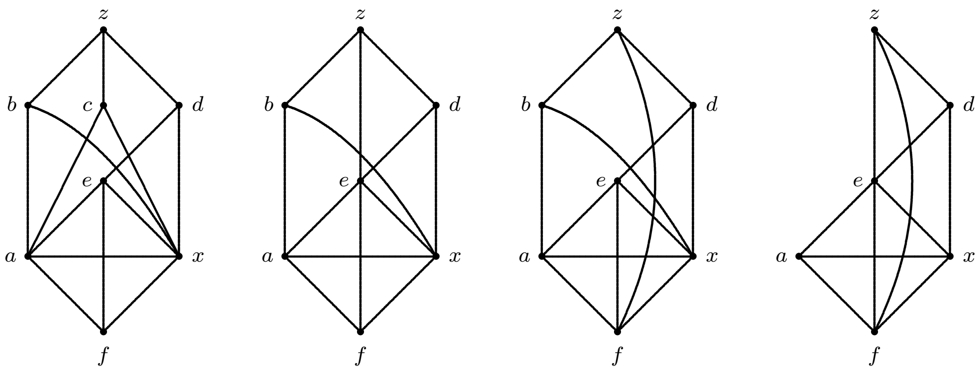


FIGURE 4. The blue 3-connected subgraphs which can arise.

So z sends at least three red edges to $\{b, c, d, e, f\}$. Since $d_R(y) = 6$, y sends at least four red edges to $\{b, c, d, e, f, z\}$. Recall that bd, cd, be, ce, bf, cf, $df \in E(R)$. Hence, $\{b, c, d, e, f, y, z\}$ induces a red subgraph with at least 14 edges. If this is not 3-connected, then by Observation 1, we have a $K_{2,3}^{(B)}$ or a $K_{1,4}^{(B)}$ in the complement. If the former, then deleting the 2-set gives a red K_5 minus at most one edge: a contradiction. If the latter, then deleting the vertex of blue degree 4 gives a red K_6 minus at most three edges. If this is not 3-connected, then we have a $K_{1,3}^{(B)}$ in the complement. Deleting the vertex of blue degree 3 then gives a red K_5: another contradiction.

Case 3. $d_{R[A]}(u) \geq 2$ for all $u \in A$.

Let $S = \{u \in A : d_{R[A]}(u) = 2\}$. We consider three sub-cases.

Case 3a. If $\{sy : s \in S\} \subset E(R)$, and $R[A \cup \{y\}]$ is not 3-connected, then $A \cup \{y\}$ contains a $K_{i,j}^{(B)}(n-4)$. Since $d_{B[A \cup \{y\}]}(u) \leq n - 6$ for all $u \in A \cup \{y\}$, we have $i \geq 2$ and $j \geq 3$. If y is in the $K_{i,j}^{(B)}(n-4)$, it must be in the j-class. Deleting y if necessary, and adding x, we have a blue 3-connected subgraph on at least $n-4$ vertices.

Case 3b. Let $\{s \in S : sy \in E(B)\} = \{a\}$. Let b, $c \in A$ be the two red neighbours of a. If $R[(A \setminus \{a\}) \cup \{y\}]$ has minimum degree at least 3, then we are done by a similar argument as in Case 3a. Indeed, if $R[(A \setminus \{a\}) \cup \{y\}]$ is not 3-connected, then we get a $K_{i,j}^{(B)}(n-5)$ in $(A \setminus \{a\}) \cup \{y\}$. Since $d_{B[(A \setminus \{a\}) \cup \{y\}]}(u) \leq n-7$ for all $u \in (A \setminus \{a\}) \cup \{y\}$, we have $j \geq i \geq 2$. It follows that y cannot belong to the $K_{i,j}^{(B)}(n-5)$. Uniting the $K_{i,j}^{(B)}(n-5)$ with x then gives a blue 3-connected subgraph on $n-4$ vertices.

So, assume without loss of generality that $d_{R[A]}(b) = 3$ and $by \in E(B)$. Let d, $e \in A$ be the other two red neighbours of b. We now make the following observation.

Observation 5. *For $n \geq 9$, let W be a set with $n-5$ vertices such that only one or two of its vertices can send red edges to $\{u, v\}$. If another vertex w sends blue edges to all of $W \cup \{u, v\}$, then we have a blue 3-connected subgraph on at least $n-4$ vertices.*

Proof. Indeed, deleting all the red neighbours of u and v in W, we have a blue $P_2 + E_{n-6}$ or a blue $P_2 + E_{n-7}$. □

By Observation 5, we must have $c \neq d$ and $c \neq e$. If c sends a blue edge to $A \setminus \{a, b, c, d, e\}$, then we have a blue 3-connected subgraph on $(A \setminus \{d, e\}) \cup \{x\}$. Similarly for d and e. So c, d, e send only red edges to $A \setminus \{a, b, c, d, e\}$. If $n \geq 10$, we have a $K_{3,n-7}^{(R)}$ with classes $\{c, d, e\}$ and $(A \setminus \{a, b, c, d, e\}) \cup \{y\}$. If $n = 9$, let $\{f\} = A \setminus \{a, b, c, d, e\}$. If $\{c, d, e\}$ has a red P_2, then $\{c, d, e, f, y\}$ has a red wheel on 5 vertices. So $\{c, d, e\}$ has a blue P_2. But then we have a blue 3-connected subgraph on $\{a, b, c, d, e, f, x\}$.

Finally, assume without loss of generality that $d_{R[A]}(b) = 2$. But then we are instantly done by Observation 5.

Case 3c. Let $\{s \in S : sy \in E(B)\} = \{a, b\}$. If either $ab \in E(R)$ or the red neighbourhoods of a and b in $A \setminus \{a, b\}$ coincide, then we are done by Observation 5. So, suppose that c, d are the red neighbours of a, and e, f are the red neighbours of b, where $\{c, d, e, f\} \subset A \setminus \{a, b\}$.

If $c = e$ and $d \neq f$, then by a similar argument as in Case 3b, we are done if d or f sends a blue edge to $A \setminus \{a, b, c, d, f\}$, or if c sends two blue edges to $A \setminus \{a, b, c, d, f\}$. If c, d, f all send only red edges to $A \setminus \{a, b, c, d, f\}$, the same argument as in Case 3b again shows that $\{c, d, f\}$ must contain a blue P_2, but this time, we have a blue 3-connected subgraph on at least six vertices of $A \cup \{x\}$. So c sends exactly one blue edge to $A \setminus \{a, b, c, d, f\}$. If $n \geq 11$, then we have a red 3-connected subgraph on $(A \setminus \{a, b\}) \cup \{y\}$. Let $n = 9$ or 10, and let $g \in A \setminus \{a, b, c, d, f\}$ be the blue neighbour of c. If $cd \in E(B)$, then we have a blue 3-connected subgraph on $\{a, b, c, d, g, x\}$. Similarly if $cf \in E(B)$. So cd, $cf \in E(R)$. But then we have a red 3-connected subgraph on $(A \setminus \{a, b\}) \cup \{y\}$.

Finally, let c, d, e, f be mutually distinct. If $n = 9$, then, since $R(C_4, P_2) = 4$, it is easy to check that we either have a red wheel on $\{c, d, e, f, y\}$, or a blue 3-connected subgraph on a subset of $\{a, b, c, d, e, f, x\}$ with six vertices. So let $n \geq 10$. If $\{c, d, e, f\}$ send only red edges to $A \setminus \{a, b, c, d, e, f\}$, then we have a $K^{(R)}_{4,n-8}$, so we are done if $n \geq 11$. If $n = 10$, we are again done by a similar argument to the fact that $\{c, d, e, f\}$ has either a red C_4 or a blue P_2. Now, if at least two of $\{c, d, e, f\}$ send a blue edge to $A \setminus \{a, b, c, d, e, f\}$, then deleting the other two of $\{c, d, e, f\}$, and uniting with $(A \setminus \{c, d, e, f\}) \cup \{x\}$, we have a blue 3-connected subgraph on $n - 4$ vertices. So, assume without loss of generality that only c out of $\{c, d, e, f\}$ sends at least one blue edge to $A \setminus \{a, b, c, d, e, f\}$. If $cd \in E(B)$, we have a blue 3-connected subgraph on $(A \setminus \{e, f\}) \cup \{x\}$. Similarly if $ce \in E(B)$ and if $cf \in E(B)$. So, cd, ce, $cf \in E(R)$. But then we have a red 3-connected subgraph on $(A \setminus \{a, b\}) \cup \{y\}$.

The proof of Theorem 2 is now complete. □

4. Further Problems

We can now quite easily see an obvious further extension to this problem. We can ask: Whenever we r-colour the edges of K_n, on how many vertices are we guaranteed to be able to find a k-connected subgraph, using s of the r colours? This question was originally asked by Bollobás. The answer is difficult to determine. For partial results to this problem and other related problems, see [3], [4] and [5].

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