Thresholds for Roman domination

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Abstract

Define a Roman dominating function (RDF) of a graph G to be a function $f : V(G) \to \{0,1,2\}$ such that every u with $f(u) = 0$ has a neighbor v with $f(v) = 2$. The *weight* of f, $w(f)$, is $\sum_{v \in V(G)} f(v)$. The Roman domination number of G, $\gamma_R(G)$, is the minimum weight of an RDF of G. It is easy to see that $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$, where $\gamma(G)$ is the domination number of G. In this paper, we determine probability thresholds for the events $\gamma_R(G) = (1 + o(1))\gamma(G)$ and $\gamma_R(G) = (2 - o(1))\gamma(G)$ in the random graph model $\mathbb{G}_{n,p}$.

1 Introduction

Throughout the paper, $\mathbb{G}_{n,p}$ denotes the probability space of all graphs on n labelled vertices, where each of the $\binom{n}{2}$ edges appears randomly and independently with probability $p = p(n)$. We use $\mathbb{P}$ and $\mathbb{E}$ to denote probability and expectation. Furthermore, for $A \subset \mathbb{G}_{n,p}$, $\mathbb{E}_A(X)$ is the expectation of X over the set A. If $\mathbb{P}(E) \to 1$ as $n \to \infty$, we say the event E happens with high probability (alternatively "almost always" or "for almost every G"), abbreviated w.h.p. For two functions $f(n)$ and $g(n)$, we say $f \sim g$ if f/g tends to 1 as n tends to ∞. We use $o(g)$ to denote a function f such that $f/g \to 0$ as $n \to \infty$. As a convention, $q = 1 - p$ and $w = pn$.

The Roman domination number of a graph G, $\gamma_R(G)$, is an invariant introduced in [3] and further studied in [4, 5, 6]. A Roman dominating

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function (RDF) of G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex in $f^{-1}(0)$ has a neighbor in $f^{-1}(2)$. The weight of f, $w(f)$, is $\sum_{v \in V(G)} f(v)$. By definition, $\gamma_R(G)$ is the minimum weight of an RDF of G. For an RDF f and $i \in \{0, 1, 2\}$, we will write $V_i(f)$ for $f^{-1}(i)$. We suppress the dependence on f when the context is clear.

For any RDF, $V_1 \cup V_2$ is a dominating set of G, so

$$w(f) = |V_1| + 2|V_2| \geq \gamma(G).$$

On the other hand, the RDF which gives weight 2 to every vertex of a minimum dominating set shows $\gamma_R(G) \leq 2\gamma(G)$. The goal of this paper is to determine when these bounds are achieved (asymptotically) for a random graph in $\mathbb{G}_{n,p}$.

Corollary 1 tells us that $\gamma_R(G) \geq 2\gamma(G) - 1$ if G has diameter 2. It is known ([2], Corollary 10.11) that if $p^2 n - 2\log n \to \infty$, then G has diameter 2. Therefore the probability threshold for the event $\gamma_R(G) = (2 - o(1))\gamma(G)$ should be no larger than $\sqrt{2\log n/n}$; the actual threshold is in fact much lower.

As it turns out, the number of isolated vertices guides the ratio γ_R/γ: when there are many $(n - o(n))$, the ratio is nearly 1, and when there are few $(o(n))$, the ratio is close to 2. This constitutes our main result.

Theorem 1 *In the probability space $\mathbb{G}_{n,p}$, if $p < 1/2$, then w.h.p.,*
(1) if $w \to 0$, then $\gamma_R(G) = (1 + o(1))\gamma(G)$
(2) if $w \to \infty$, then $\gamma_R(G) = (2 - o(1))\gamma(G)$
(3) if $w \to c \in \mathbb{R}$, then there is a constant $c' \in (1, 2)$ such that $\gamma_R(G) = (c' + o(1))\gamma(G)$.

The reason we insist $p < 1/2$ in Theorem 1 is that p must be bounded away from 1 to ensure that $\gamma(G)$ is (almost always) unbounded, and 1/2 is simply convenient. The Theorem would remain true if we stipulated that $p < b$ for any $b < 1$, or even $p \to 1$ slowly. The only modification needed would be in the bound on $q^{\alpha n}$ in the second paragraph of the Proof of Theorem 1. In fact, based on the discussion above about graphs of diameter 2, we are justified in requiring $p \to 0$; we will therefore assume this through the rest of the paper.

The paper is organized as follows: in Section 2, we develop lemmas we will need for the proof of Theorem 1, which follows in Section 3.

For further background and notation, see [7] for graph theory and [1] for probability.

2 Some Lemmas

In this section we prove several lemmas we will need in the next Section. We begin with two on degrees in random graphs.

In the remainder of the paper, let $X_k(G)$ be the random variable counting the vertices of degree k in G. A simple first moment/second moment argument shows:

Lemma 1 *If $p \to 0$, then, for almost every $G \in \mathbb{G}_{n,p}$,*

$$X_0(G) \sim \mathbb{E}(X_0) \sim \frac{n}{e^w}.$$

The next lemma tells when very few vertices have degree significantly less than the expected degree ($\approx w$).

Lemma 2 *Suppose $p \to 0$ and $w \to \infty$. Then almost every $G \in \mathbb{G}_{n,p}$ has at most $(9/10)^w n$ vertices of degree less than $w/2$.*

Proof. Choose n large enough that $p \leq 1/100$. Now, the number of vertices of degree less than k is $\sum_{i=0}^{k-1} X_i$; call this quantity Y_k. Then

$$\mathbb{E}(Y_{w/2}) = n \sum_{i=0}^{w/2-1} \binom{n}{i} p^i (1-p)^{n-1-i} \leq n \sum_{i=0}^{w/2-1} \frac{w^i \, e^{p(i+1)}}{i! \, e^w}$$

$$\leq (1+o(1))\, n \, \frac{w}{2} \, \frac{w^{w/2} e^{101w/200}}{(w/2)^{w/2} e^w} < \frac{(1+o(1))\, w \, (.863)^w}{2} n < .87^w \, n$$

if n is large enough. By the Chernoff bound, the probability that G has at least $(9/10)^w \, n$ vertices of degree less than $w/2$ is at most $(.87/.9)^w \to 0$. $\square$

We now turn our attention to Roman domination. In particular, we try to relate $\gamma_R(G)$ to properties of the degrees of vertices of G. Our first lemma deals with optimal RDFs; its corollaries will prove quite useful.

Lemma 3 *Let G be a graph. Then G has a minimum weight RDF such that*
1) V_1 is an independent set,
2) each vertex in V_0 has at most 1 neighbor in V_1,
3) $E_G[V_1, V_2] = \emptyset$.

Proof. Let f be any minimum-weight RDF of G. If $G[V_1]$ contains edge uv, move u to V_2 and v to V_0. If some vertex $u \in V_0$ has neighbors v and w in V_1, move u to V_2 and v and w to V_0. If $u \in V_1$, $v \in V_2$, and $uv \in E(G)$, then move u to V_0. None of these moves increase the weight of f. Since each move decreases $|V_1|$, only a finite sequence of moves is possible. When no more such move is left, we have the claimed RDF. □

Corollary 1 *Let G be a graph of diameter 2. Then $\gamma_R(G) \geq 2\gamma(G) - 1$.*

Proof. Let f be a minimum weight RDF of G as claimed in Lemma 3. Suppose $|V_1| \geq 2$, and let $u, v \in V_1$ $(u \neq v)$. Since G has diameter 2, u and v have a common neighbor w. Now $w \notin V_1$ by (1), $w \notin V_0$ by (2), and $w \notin V_2$ by (3). Therefore, $|V_1| \leq 1$. Since $|V_1 \cup V_2| \geq \gamma(G)$, $w(f) = |V_1| + 2|V_2| \geq 2\gamma(G) - 1$. □

Corollary 2 *Let G be a graph with at most m vertices of degree less than $k \geq 1$. Then*

$$\gamma_R(G) \geq 2\gamma(G) - \frac{n}{k+1} - m.$$

Proof. Let f be a minimum weight RDF of G guaranteed by Lemma 3. There are at least $|V_1| - m$ vertices of V_1 with degree at least k, and since their neighborhoods are disjoint and contained in V_0, we see

$$k\left(|V_1| - m\right) \leq |V_0| \leq n - |V_1|,$$

and so $|V_1| \leq n/(k+1) + m$. Since $|V_1 \cup V_2| \geq \gamma(G)$ and $\gamma_R(G) = 2|V_2| + |V_1| \geq 2\gamma(G) - |V_1|$, the result is immediate. □

3 The Main Result

We are now ready to prove Theorem 1

Proof. Suppose $w \to 0$. By Lemma 1, w.h.p., $X_0 \sim n$. All vertices of X_0 are in any dominating set, so almost always

$$(1 - o(1))n = X_0 \leq \gamma(G) \leq \gamma_R(G) \leq n,$$

showing that $\gamma_R(G) = (1+o(1))\gamma(G)$. This establishes (1).

Now consider $w \to \infty$. Let $\alpha = \ln\ln w/w$. Define Γ_r to be the number of dominating sets of $G \in \mathbb{G}_{n,p}$ with cardinality r. Then

$$\mathbb{E}(\Gamma_{\alpha n}) = \binom{n}{\alpha n}(1-q^{\alpha n})^{n-\alpha n}$$

$$\sim \left(\frac{(1-q^{\alpha n})^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)^n \frac{1+o(1)}{\sqrt{2\pi\alpha(1-\alpha)n}}.$$

Since $\left(\sqrt{2\pi\alpha(1-\alpha)n}\right)^{-1} = o(1)$, $\mathbb{E}(\Gamma_{\alpha n}) \to 0$ if

$$(1-q^{\alpha n})^{1-\alpha} < \alpha^\alpha(1-\alpha)^{1-\alpha}$$

when n is large. Since $p < 1/2$, we have $q \geq e^{-2p}$, and so

$$1 - q^{\alpha n} \leq 1 - e^{-2p\alpha n} \leq \exp(-e^{-2p\alpha n}).$$

Hence we need only show

$$(1+o(1))\exp(-e^{-2p\alpha n}) < \alpha^\alpha(1-\alpha)^{1-\alpha}.$$

Taking logarithms and rewriting in terms of w, the LHS is asymptotically $\ln(w)^2$, while the RHS is

$$\frac{\ln\ln w}{w}\ln\left(\frac{w}{\ln\ln w}\right) + \frac{w-\ln\ln w}{w}\ln\left(\frac{w}{w-\ln\ln w}\right) < \frac{\ln(w)\ln\ln(w)}{w}$$

when w is large enough. We conclude that $\mathbb{E}(\Gamma_{\alpha n}) \to 0$, and that w.h.p. $\gamma(G) \geq (\ln\ln w/w)n$.

By Lemma 2, a.e. G has at most $(9/10)^w n$ vertices of degree less than $w/2$. Therefore we can apply Lemma 2 to see

$$\gamma_R(G) \geq 2\gamma(G) - (2/w + (9/10)^w)\, n.$$

From the last paragraph we know that for a.e. G, $(2/w + (9/10)^w)n = o(\gamma(G))$, which establishes (2).

Finally, for (3), suppose $w \to c \in \mathbb{R}$ and let $\mathcal{A} \subset \mathbb{G}_{n,p}$ be the set of graphs with a vertex of degree at least $K = \ln\ln n$. Then

$$\mathbb{P}(\mathcal{A}) \leq n\binom{n}{K}p^K \leq \frac{nw^K}{K!} \to 0.$$

Let $D_0, \ldots, D_n$ and $RD_0, \ldots, RD_n$ be the vertex-exposure martingales given by

$$D_i = \mathbb{E}_{\overline{\mathcal{A}}}\left(\gamma(G[v_1, \ldots, v_i])\right)$$
$$RD_i = \mathbb{E}_{\overline{\mathcal{A}}}\left(\gamma_R(G[v_1 \ldots, v_i])\right).$$

On $\overline{\mathcal{A}}$, $|D_{i+1} - D_i| \leq K$ and $|RD_{i+1} - RD_i| \leq 2K$, so Azuma's Inequality tells us

$$\mathbb{P}\left(D_0 - \sqrt{n \ln^3 n} \leq \gamma(G) \leq D_0 + \sqrt{n \ln^3 n}\right) \geq 1 - \frac{1}{\sqrt{n}}$$

$$\mathbb{P}\left(RD_0 - 2\sqrt{n \ln^3 n} \leq \gamma_R(G) \leq RD_0 + 2\sqrt{n \ln^3 n}\right) \geq 1 - \frac{1}{\sqrt{n}}.$$

Since $\gamma(G)$ and $\gamma_R(G)$ are at least $n/\ln\ln n$ on $\overline{\mathcal{A}}$, we see that w.h.p. $\gamma(G) \sim D_0$ and $\gamma_R(G) \sim RD_0$. Let $c' = RD_0/D_0$. To establish (3), it remains only to show $c' \in (1, 2)$.

By Lemma 1, $X_0 \sim n/e^w$, so

$$\gamma_R(G) \leq 2\gamma(G) - \frac{n + o(n)}{e^w} \leq \left(2 - \frac{1 + o(1)}{e^w}\right)\gamma(G),$$

and $c' < 2$. Let Z be the number of triples $x, y, z \in V(G)$ such that x and z are adjacent to y and no other edges contain x, y, or z. A simple first moment/second moment argument shows $Z \sim nw^2/2e^{3w}$. Any optimal weight RDF puts weight 2 on the central vertex of each triple counted by Z, so $|V_2| \geq Z$, and

$$\gamma_R(G) \geq \gamma(G) + Z \geq \gamma(G) + \frac{(1 + o(1))w^2}{2e^{3w}} n \geq \left(1 + \frac{(1 + o(1))w^2}{2e^{3w}}\right)\gamma(G),$$

so $c' > 1$. This completes the proof. □

References

[1] N. Alon and J. Spencer. *The Probabilistic Method (2nd ed.)*. New York: John Wiley & Sons 2000.

[2] B. Bollobás. *Random Graphs (2nd ed.)*. Cambridge: Cambridge University Press 2001.

[3] E.J. Cockayne, P.A. Dreyer, S.M. Hedetniemi, and S.T. Hedetniemi. *Roman domination in graphs*. Disc. Math. 278 (2004) 11-22.

[4] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae. *The algorithmic complexity of Roman domination,* submitted.

[5] E.W. Chambers, W. Kinnersley, N. Prince, and D.B. West. *Extremal Problems for Roman domination,* submitted.

[6] M. A. Henning. *A characterization of Roman trees.* Discuss. Math. Graph Theory 22 (2) (2002) 325-334.

[7] D.B. West. *Introduction to Graph Theory, 2nd ed.* Upper Saddle River: Prentice Hall, 2001.