Math 231/199: Calculus 2 Merit

Worksheet 22

December 4, 2007

(1) Use the integral test to show the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

(2) Use the integral test remainder estimate

$$S - S_n \leq \int_{n+1}^{\infty} f(x)\,\mathrm{d}x$$

to approximate $\sum_{k=1}^{\infty} \frac{1}{k^2}$ to within 1/10.

(3) Let

$$I_n = \int_0^{\pi/2} \cos^{2n}(x)\,\mathrm{d}x.$$

Prove the reduction formula $I_n = \frac{2n-1}{2n}\, I_{n-1}$.

(4) Use your answer to the last question to show that

$$I_n = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \frac{\pi}{2}.$$

(5) Use the formula for I_n in the last question to show

$$I_n = \frac{(2n)!}{4^n n!^2} \frac{\pi}{2}.$$

(6) Define

$$J_n = \int_0^{\pi/2} x^2 \cos^{2n}(x)\,\mathrm{d}x.$$

Use integration by parts twice to show that

$$I_n = n(2n-1)J_{n-1} - 2n^2 J_n.$$

(HINT: Let $\mathrm{d}v$ be the power of x both times.)

(7) Use your answers to the last two questions to write

$$\frac{\pi}{4n^2} = \frac{4^{n-1}(n-1)!^2}{(2n-2)!} J_{n-1} - \frac{4^n n!^2}{(2n)!} J_n.$$

(8) Sum the formula of the last question from $n = 1$ to $n = N$ to get

$$\frac{\pi}{4} \sum_{n=1}^{N} \frac{1}{n^2} = J_0 - \frac{4^N N!^2}{(2N)!} J_N.$$

(9) Show that $J_0 = \pi^3/24$.

(10) Show that $x < \frac{\pi}{2} \sin(x)$ for $0 < x < \pi/2$.

(11) Use the inequality of the last question along with the comparison test to show that

$$J_N < \frac{\pi^2}{4}(I_N - I_{N+1}) = \frac{\pi^2}{8}\frac{I_N}{N+1}.$$

(12) Use the result of the last question to show that

$$0 < \frac{4^N N!^2}{(2N)!} J_N < \frac{\pi^3}{16(N+1)}.$$

(13) Use the Squeeze Law (or Sandwich Theorem) to show that

$$\lim_{N \to \infty} \frac{4^N N!^2}{(2N)!} J_N = 0.$$

(14) Take the limit as $N \to \infty$ of both sides of the equation of Question 8 to find $\sum_{k=1}^{\infty} 1/k^2$.

(15) Wasn't that great?