Math 231/199: Calculus 2 Merit

Worksheet 8

September 20 & 27, 2007

(1) Write out the first 8 terms of the following sequences:

$$a_n = \frac{1}{n^2+1}$$

$$a_n = n!$$

$$a_n = (-2)^n$$

$$a_n = \cos(\pi n)$$

$$a_n = \sin(\pi n)$$

(2) Find the limits of the following sequences:

$$a_n = \frac{1}{n^3}$$

$$a_n = \frac{2n}{n+1}$$

$$a_n = \frac{n^2 - n}{n^{3/2}}$$

(3) Determine if the following sequences converge, and find their limit if they do:

$$a_n = \frac{3n^2 + 1}{2n^2 - 1}$$

$$a_n = \frac{n^2 + 1}{n^3 + 1}$$

$$a_n = \frac{\cos(n)}{e^n}$$

$$a_n = (-1)^n \frac{n + 2}{n^2 + 4}$$

$$a_n = \frac{n 2^n}{3^n}$$

$$a_n = ne^{-n}$$

$$a_n = \sqrt{n^2 + n} - n$$

$$a_n = \ln(2n + 1) - \ln(n)$$

$$a_n = \left| \cos\left(\frac{\pi n}{2}\right) + \sin\left(\frac{\pi n}{2}\right) \right| \frac{2n-1}{n+1}$$

(4) Let a be a positive real number. Show that $\lim_{n\to\infty} \frac{\ln(n)}{n^a} = 0$. Conclude that, if n is large enough, $\ln(n) < n^a$ (this is a Very Important Fact).

(5) Show

$$\lim_{n\to\infty} \frac{n\ln(n)}{n^{4/3}+1} = 0.$$

(6) Find a sequence $\{a_n\}_{n=1}^{\infty}$ and a function $f(x)$ such that $\lim_{n\to\infty} a_n = L$ but $\lim_{n\to\infty} f(a_n) \neq f(L)$. What does this say about the Substitution Rule?

(7) Let

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Find $\lim_{n\to\infty} a_n$. (HINT: define $b_n = \ln(a_n)$, and start by finding $\lim_{n\to\infty} b_n$.)

(8) Find all values of p such that $\lim_{n\to\infty} \frac{1}{p^n}$ converges.

(9) Find all values of p such that $\lim_{n\to\infty} \frac{1}{n^p}$ converges.

(10) Determine if the following sequences are monotone:

$$a_n = \frac{n-1}{n+1}$$

$$a_n = \frac{e^n}{n}$$

$$a_n = \frac{n!}{5^n}$$

$$a_n = \frac{n\ln(n)}{n^2 + \sqrt{n}}$$

(11) Determine if the following sequences are bounded:

$$a_n = \frac{3n^2 - 2}{n^2 + 1}$$

$$a_n = \sin\left(n^{n^n}\right)$$

$$a_n = e^{\cos(n)}$$

$$a_n = n \sin(n)$$

(12) The world population in 1960 was approximately 3.049 billion. Start with $a_0 = 3.049$ and define $a_{n+1} = a_n + .005a_n^{2.01}$. Then a_n approximates the world population in the year 1960+n. Find a_{10}, a_{20}, and a_{30} and compare these to the world populations in 1970 (3.721 billion), 1980 (4.473 billion), and 1990 (5.333 billion), respectively. Then find a_{75}. What does this say to you about the year 2035?

(13) Let $a_1 = \sqrt{2}$ and define $a_{n+1} = \sqrt{2 + a_n}$.

(a) Assuming that $\lim_{n\to\infty} a_n$ exists, find its value (HINT: take the limit of both sides of the formula defining a_{n+1}).

(b) Show by induction that, for all n, $1 \leq a_n \leq 2$.

(c) Show by induction that, for all n, $a_{n+1} \geq a_n$.

(d) Conclude from (b) and (c) that $\lim_{n\to\infty} a_n$ exists, and so it equals your answer from (a).