Sequences and Series Review

Two very important (distinct) mathematical objects are *sequences* and *series*. A sequence is an infinite *list* of numbers, whereas a series is an infinite *sum*. With a series $\sum_{n=1}^{\infty} a_n$, though, we can associate two sequences:
1) its sequence of partial sums (S_n), where $S_n = \sum_{k=1}^{n} a_k$
2) its sequence of terms (a_n).

Our primary concern with both sequences and series is *convergence*. A sequence (a_n) converges if $\lim_{n\to\infty} a_n = L$ for some real number L. A series converges if $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = L$ for some real number L; this is the same as saying the sequence of partial sums (S_n) converges. If a sequence or series does not converge, we say it diverges.

We have developed many tools for determining whether a given sequence or series converges. The ultimate goal of all these tests is applying them to Taylor series. We know that if the Taylor series of f at a converges for a given value of x, then it converges to $f(x)$. This is why we only test for convergence rather than try to evaluate these series: once we have convergence, we know right away what it converges to.

	Sequence	Series
What it is	list	sum
Written	(a_n) $a_1, a_2, a_3, \ldots$	$\sum_{n=1}^{\infty} a_n$ $a_1 + a_2 + a_3 + \cdots$
Converges if	$\lim_{n\to\infty} a_n = L$	$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = L$ $\lim_{n\to\infty} S_n = L$
Tests	Limit Laws, Substitution Law Squeeze Law Limits of Functions and Sequences Bounded and Monotone L'Hôpital's Rule	nth Term Test Integral Test Comparison Test Limit Comparison Test Alternating Series Test Ratio Test Root Test p-Series Geometric Series

1 Sequences

Limit Laws, Substitution Law

These laws tell us how to compute limits of sequences. If we can determine what a sequence converges to, then trivially it converges. Recall the Substitution Law says if
1) $\lim_{n\to\infty} a_n = L$
2) f is continuous at L,
then $\lim_{n\to\infty} f(a_n) = f(L)$.

Squeeze Law

The Squeeze Law is very intuitive: if $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also. Often we apply this with sin and cos, bounding them between -1 and 1.

Limits of Functions and Sequences

This is the idea that if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$ also. For example, $\lim_{x\to\infty} 2 + 1/x = 2$, so $\lim_{n\to\infty} 2 + 1/n = 2$.

This rule essentially exists to justify using L'Hôpital's Rule on sequences.

Bounded and Monotone

By a theorem, any sequence (a_n) which is bounded (for some positive number M, and for all n, $-M \leq a_n \leq M$) and monotone (for all large enough n, either the a_n's are increasing or decreasing) converges.

This theorem can only tell us *that* a sequence converges, not *to what* it converges.

L'Hôpital's Rule

L'Hôpital's Rule says that if $\lim_{n\to\infty} f(n)$ and $\lim_{n\to\infty} g(n)$ are both 0 or both ∞, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

Using standard techniques, we can use L'Hôpital's Rule to determine limits of indeterminate forms $0 \cdot \infty$, 0^∞, 1^∞, and ∞^0.

Test	Sequence	Check	Limit
Limit Laws	(a_n)	$\lim a_n = L$	L
Substitution Law	$(f(a_n))$	$\lim a_n = L$ f continuous at L	$f(L)$
Squeeze Law	(b_n)	$a_n \leq b_n \leq c_n$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$	L
Functions and Sequences	$(f(n))$	$\lim_{x\to\infty} f(x) = L$	L
Bounded and Monotone	(a_n)	$-M \leq a_n \leq M$ for all n a_n monotone	No info
L'Hôpital's Rule	$(f(n)/g(n))$	$\lim f(n) = \lim g(n) = 0$ or $\lim f(n) = \lim g(n) = \infty$	$\lim f'(n)/g'(n)$

2 Series

nth Term Test

If $\sum_{n=1}^{\infty} a_n$ converges, to L say, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0.$$

The nth Term Test is the contrapositive of this statement: if $\lim_{n\to\infty} a_n$ does not exist or is not 0, then $\sum_{n=1}^{\infty} a_n$ diverges.

This test is easy to apply, and should probably be the first test applied to determine if a certain series converges or diverges.

Integral Test

If $\sum_{n=1}^{\infty} a_n$ is a series and $f(x)$ is a function such that
(1) $f(x)$ is positive, decreasing, and continuous
(2) $f(n) = a_n$ for all n,
then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,\mathrm{d}x$ both converge or both diverge.

Comparison Test

The Comparison Test says that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive-term series,
(1) if $a_n \leq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
(2) if $a_n \geq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

How you use the Comparison Test depends on whether you think $\sum_{n=1}^{\infty} a_n$ converges or diverges. If you think it converges, try to find a convergent series $\sum_{n=1}^{\infty} b_n$ with $a_n \leq b_n$, and if you think it diverges, find a divergent series $\sum_{n=1}^{\infty} b_n$ with $a_n \geq b_n$.

Limit Comparison Test

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive-term series and $\lim_{n\to\infty} a_n/b_n = L$ with L a positive real number (note $L \neq 0$), then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

Alternating Series Test

If $\sum_{n=1}^{\infty} a_n$ is an alternating series (the terms a_n alternate sign) and
(1) $\lim_{n\to\infty} a_n = 0$
(2) $|a_n|$ is decreasing
then $\sum_{n=1}^{\infty} a_n$ converges.

The alternating series test comes with a remainder estimate. If $\sum_{n=1}^{\infty} a_n$ is an alternating series converging to L, then

$$|L - \sum_{n=1}^{N} a_n| \leq |a_{N+1}|.$$

Using this remainder estimate, it is easy to approximate the value of an alternating series to any degree of accuracy: if you want to know the value to within an error of E, find an N such that $|a_{N+1}| \leq E$, and then simply sum the first N terms.

Geometric Series

A geometric series is a series $\sum_{n=1}^{\infty} a_n$ such that $a_n = a_1 r^{n-1}$ for some real number r. Not only do we know when these converge, but we also have a formula for their value:
(1) if $|r| < 1$, $\sum_{n=1}^{\infty} a_n = a_1/(1-r)$
(2) if $|r| \geq 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

Ratio Test

The Ratio Test says that if $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$, then
(1) if $\rho < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely
(2) if $\rho > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
The test is inconclusive if $\rho = 1$.

The Ratio Test allows us to say $\sum_{n=1}^{\infty} a_n$ is almost a geometric series with ratio ρ, since for a geometric series $\rho = |r|$. This is why the convergence criteria for Ratio Test and geometric series are similar.

The Ratio Test is particularly suited for series involving factorials and powers, since most of it will cancel when computing the limit.

Root Test

The Root Test says if $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, then the conclusions of the Ratio Test hold.

The Ratio Test and the Root Test always give the same value of ρ.

When using the Root Test, one often has to compute a limit such as $\lim_{n\to\infty} \sqrt[n]{n}$. The secret for these sort of limits is to convert them to powers of e:

$$\sqrt[n]{n} = n^{1/n} = (e^{\ln n})^{1/n} = e^{\ln n/n}.$$

Since $\lim_{n\to\infty} \ln n/n = 0$, the Substitution Rule says $\lim \sqrt[n]{n} = e^0 = 1$.

p-Series

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. This is a consequence of the Integral Test.

Test	Check	Conclude
nth Term	$\lim_{n\to\infty} a_n \neq 0$	Diverges
Integral	$f(x) > 0$, cont., decr.	
	$f(n) = a_n$	
	if $\int_1^\infty f(x)\mathrm{d}x$ converges	Converges
	if $\int_1^\infty f(x)\mathrm{d}x$ diverges	Diverges
Comparison	$a_n \leq b_n$	
	$\sum_{n=1}^\infty b_n$ converges	Converges
	$a_n \geq b_n$	
	$\sum_{n=1}^\infty b_n$ diverges	Diverges
Limit Comparison	$\lim_{n\to\infty} a_n/b_n = L$, $L \neq 0$	
	$\sum_{n=1}^\infty b_n$ converges	Converges
	$\sum_{n=1}^\infty b_n$ diverges	Diverges
Alternating Series	$\lvert a_n \rvert$ decreases	
	$\lim_{n\to\infty} a_n = 0$	Converges
Geometric	$\lvert r \rvert < 1$	Converges to $a_1/(1-r)$
	$\lvert r \rvert \geq 1$	Diverges
Ratio	$\rho < 1$	Converges (abs.)
	$\rho > 1$	Diverges
Root	$\rho < 1$	Converges (abs.)
	$\rho > 1$	Diverges
p-Series	$p > 1$	Converges
	$p \leq 1$	Diverges

3 Taylor series

The *Taylor series* for a function $f(x)$ at a point a is simply

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

The Maclaurin series is just the Taylor series with $a = 0$. An important feature of the Taylor series is that if it converges for a given value of x, then it converges to $f(x)$.

To compute the Taylor series for $f(x)$ at a, we just have to find the derivatives of f, evaluate them at a, and plug into the formula.

Taylor series have nice convergence properties. For example, they always converge on an interval (the *interval of convergence*) whose midpoint is a. The *radius of convergence* is half the length of the interval of convergence.

Related to Taylor series are *Taylor polynomials*. The nth degree Taylor polynomial of $f(x)$ at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-a)^k,$$

so $P_n(x)$ is just the first $n+1$ terms of the Taylor series. Taylor's Theorem tells us that, as long as we can differentiate $f(x)$ $n+1$ times, $f(x) - P_n(x) = R_n(x)$, where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

for some z between x and a. In general, it is impossible to solve for z here. In practice, we try to bound $R_n(x)$ by treating it as a function of z and finding where it achieves its extrema.

4 Convergence

There are two types of convergence for a series $\sum_{n=1}^{\infty} a_n$: absolute and conditional. $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. All absolutely convergent sequences converge. Conditional convergence happens when $\sum_{n=1}^{\infty} |a_n|$ diverges but $\sum_{n=1}^{\infty} a_n$ converges.

The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^n/n$ converges conditionally, whereas $\sum_{n=1}^{\infty}(-2/3)^n$ converges absolutely.

5 Problems

1. Determine the limits of the following sequences: (1) $a_n = \ln n/n$ (2) $a_n = (4/3)^n$ (3) $a_n = \sqrt[n]{n^{\ln n}}$ (4) $a_n = (2/3)^n n^2$ (5) $a_n = \frac{1{,}000{,}000^n}{n!}$ (6) $a_n = \frac{\sin n}{\ln\ln\ln(n)}$ (7) $a_n = \frac{1}{\sqrt{n}}$.

2. Show that the sequence $\sqrt{25 - 1/\sqrt{n}}$ converges by first applying the bounded and monotone test, then determine its limit.

3. Does the series $\sum_{n=1}^{\infty}(-1/2)^n + 1/n$ converge?

4. Does the series $\sum_{n=1}^{\infty} \frac{1}{n \ln n \ln\ln n}$ converge? Determine all values of p for which $\sum_{n=1}^{\infty} \frac{1}{n \ln n (\ln\ln n)^p}$ converges.

5. Determine if the following series converge or diverge.

(1)

$$\sum_{n=1}^{\infty} \frac{1}{\ln n}$$

(2)

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{3}\right)^n$$

(3)

$$\sum_{n=1}^{\infty} \frac{|\sin(n^4 \ln(\cos^2(\sqrt{n})) + 1)|}{n^{17/16}}$$

(4)

$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$$

(5)

$$\sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n}}$$

(6)

$$\sum_{n=1}^{\infty} \frac{3^n n^{1000}}{10^n}$$

(7)

$$\sum_{n=1}^{\infty} (2/3)^n - (3/5)^n.$$

6. Use the Limit Comparison Test to determine if the following series converge or diverge.

(1)

$$\sum_{n=1}^{\infty} \frac{4\sqrt{n} + 78}{13n^2 + 42}$$

(2)

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$

(3)

$$\sum_{n=1}^{\infty} \frac{4^n + 16^{n/2}}{7^n}$$

(4) $\sum_{n=1}^{\infty} 1 - 1/n$.

7. Determine if the following series converge or diverge.

(1) $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$

(2) $\sum_{n=1}^{\infty}(-1)^n - 1/n$
(3) $\sum_{n=1}^{\infty} \sqrt[n]{(-1)^n n^2}$.

8. Determine the value of the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^n/n$ to within .2.

9. Find the Taylor series of $\ln(1+x)$ at $a = 0$ and determine its interval of convergence.

10. Show that the Taylor series for e^x, $\sin(x)$, and $\cos(x)$ converge for all real x.

11. Determine the interval and radius of convergence of the following series
(1)

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n}$$

(2)

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n!}$$

(3)

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

(4)

$$\sum_{n=1}^{\infty} \frac{(-2)^n + 3^n}{7^n} x^n.$$

12. Use your answer from Problem 9 to find $\ln(3/2)$ accurate to 3 decimal places.

13. Use L'Hôpital's Rule to find $\lim_{x \to 0}(x - \sin(x))/x^3$. Then find the Taylor series of $(x - \sin(x))/x^3$. What is its limit as x goes to 0?

14. Find the fourth degree Taylor polynomial of $\sin(x)$ at $a = 0$ and also the remainder estimate. How many decimal places of accuracy can you guarantee if you approximate $\sin(1/2)$ with $P_4(1/2)$?

15. Suppose the Maclaurin series for $f(x)$ has interval of convergence $(-2, 2]$. What is the radius of convergence of the Maclaurin series of $f(3x)$?

16. Without calculating derivatives, find the Taylor series of the following functions at $a = 0$.
(1) e^{5x^2} (2) $\sin(x^9)$ (3) $x/(1-x)^2$ (4) $(\sin(x) - x)/x^2$.

17. Show that $\int \sin(x)\mathrm{d}x = -\cos(x) + C$ using Taylor series.