Research Statement, Naeem N. Sheikh

My research interests over the course of my PhD studies have included intersection families, enumeration problems of matchings, induced Ramsey theory, and decompositions of planar graphs. For each of these interests, underneath I will describe the results I have worked on and further questions that I am exploring.

Intersection Families A typical question in this area considers a family of subsets of $[n]$ with specified properties of intersections between the subsets and then asks for the maximum possible cardinality of such a family. A famous conjecture in this area is the Snevily conjecture: let $\mathcal{F}$ be a family of subsets of $[n]$, $K = \{k_1, ..., k_r\}$ and $L = \{l_1, ..., l_s\}$ be sets of integers such that $|F_i| \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \in L$ for all $F_i, F_j \in \mathcal{F}, i \neq j$. For any K and L with min $k_i >$ max l_j, $|\mathcal{F}| \leq \binom{n}{s}$.

Snevily considers this to be a hard result to prove and offers a weaker conjecture in [1] which states that for any K and L with min $k_i >$ max l_j, $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

In my joint work with Kyung-Won Hwang [2], we have proven two results. One says that Snevily's weaker conjecture is true under the extra hypothesis that K is a set of consecutive integers. The other says that the weaker conjecture is also true under the condition that the family $\mathcal{F}$ has a non-empty center, i.e. the intersection of all the sets in $\mathcal{F}$ is non-empty. We have employed the polynomial method in proving these results.

I am interested in pursuing this area of research further especially the usage of polynomial method to extract bounds on cardinalities of intersection families under various hypotheses. In particular, trying to get a proof of the afore-mentioned Snevily's weaker conjecture, using the polynomial method.

Here is a small problem that I worked on which settled an open question: R. Graham, M. Simonovits, and V.T. Sós [3] posed the following question. Suppose S is a convex subset of $\mathbb{Z}^k$ and let $\mathcal{A}$ be a family of subsets of S such that $A_i \cap A_j$ is convex and nonempty for $A_i, A_j \in \mathcal{A}, A_i \neq A_j$. If $\mathcal{A}$ is a family of maximum cardinality with this property, is it true that $\bigcap_{A_i \in \mathcal{A}} A_i$ is also non-empty? Kyung-Won Hwang and I answered [4] this question in the negative by exhibiting a set S and a family of subsets of S for every dimension $k \geq 2$, and then posed a modified question which we think has positive answer.

Induced Ramsey Theory

The *induced Ramsey number*, $IR(G, H)$, is the greatest positive integer N such that for each graph F on at most $N-1$ vertices, there exists a 2-coloring of its edges with red and blue such that no induced copy of G in F has all its edges red and no induced copy of H in F has all its edges blue. Say that a graph F is an *IR-graph for graphs G and H*, if for each 2-coloring of edges of F with red and blue, either some induced in F subgraph isomorphic to G has all its edges colored red, or some induced in F subgraph isomorphic to H has all its edges colored blue. In these terms, the *induced Ramsey number*, $IR(G, H)$, is the least order of an IR-graph for G and H. The fact that an IR-graph exists for each G and H and thus $IR(G, H)$ is finite was proved independently by Deuber, Erdős et al., and Rödl. A characteristic similar to the induced Ramsey number is the *weak induced Ramsey number*, $IR_w(G, H)$ – the least positive integer N such that there exists an N-vertex graph F with the property that for each 2-coloring of its edges with red and blue, either the red subgraph of F contains an induced (in this red graph) copy of G, or the blue subgraph of F contains an induced (in this blue graph) copy of H.

Gorgol and Łuczak [5] gave an example of a pair of graphs for which the induced Ramsey number is greater than the weak induced Ramsey number. Namely, they showed that $IR(P_3, P_4) = 7$ and $IR_w(P_3, P_4) = 6$, where P_k is the path with k vertices. They also asked whether there exists a sequence $\{H_n\}_{n=1}^{\infty}$ of graphs such that for some graph G,

$$\lim_{n\to\infty} \sup \frac{IR(G, H_n)}{IR_w(G, H_n)} > 1. \tag{1}$$

Among other results, Gorgol and Łuczak proved that for every $n \geq 3$,

$$1.5n - 1 \leq IR(P_3, P_n) \leq 2n - 1 \quad \text{and} \quad 4n/3 \leq IR_w(P_3, P_n) \leq 5n/3.$$

The above paper served as a source of inspiration for my joint work with A. Kostochka in this area [6]. We proved that for any graph H, the induced Ramsey number $IR(P_3, H)$ is at most the total number of vertices and edges in H. This bound is shown to be sharp for disjoint collections of cliques. Then, we proved an improved upper bound in case that H is a disjoint collection of complete multipartite graphs, a bound that is also shown to be sharp.

Regarding the original number studied in the paper by Gorgol and Łuczak, namely $IR(P_3, P_n)$, we prove an improved lower bound for it, namely $3n/2 + n/32$. We also answer their question about the asymptotic difference between the induced Ramsey numbers and the weak induced Ramsey numbers in the affirmative.

Currently I am working on generalizing the results for $IR(P_3, H)$ to $IR(K_{1,t}, H)$ (since P_3 is isomorphic to $K_{1,t}$).

Decompositions of Planar Graphs

A *decomposition* of a graph G is a set of pair-wise edge-disjoint subgraphs of G whose union is G. There are many famous families of decomposition problems that have attracted considerable attention of researchers. Borodin, Ivanova, Kostochka, G. Yu, and I worked on a number of results having to do with decompositions of planar graphs under various cycle structure conditions into a forest and another graph whose maximum degree is not too high. In particular, we have proven that a planar graph with girth at least nine can be decomposed into a forest and a matching. Graphs of girth 7 can not always be decomposed into a forest and a matching, and it remains an open question whether this is always true for graphs of girth 8 or higher. Then, we proved a characterization of a broader class of planar graphs that can be decomposed into a forest and a matching: in such graphs, every cycle with at least 4 edges has at least 9 non-triangular edges (i.e. edges that are not in any triangle). Note that such graphs can have $3-$cycles. Another major result we proved is that a planar graph without $4-$cycles can be decomposed into a forest and a graph with maximum degree at most 5. It is known that this degree of the non-forest graph in the decomposition can not be improved to 3, but it is not known whether it can be improved to 4.

In relation to this work, we have also proven that the minimax degree of a planar graph without $4-$cycles is at most 7 and this bound is sharp. The *minimax* degree of a graph is the minimum of the maximum degree of an edge over all edges, where the maximum degree of an edge is the maximum of the degrees of its two incident vertices.

The open questions mentioned above are directions for my future work.

Enumerations of Matchings

Let G_n be the planar graph obtained by taking the triangular lattice graph with $n+1$ rows of vertices (n rows of triangles) and then in each triangular face "facing up", put an extra vertex and make it adjacent to the 3 vertices of the enclosing face. Propp [7] conjectured that for G_n, the number of (perfect) matchings is divisible by 3. In joint work with Stephen Hartke and Kyung-won Hwang [8], we have shown that $3^{(n+1)/2}$ divides the number of matchings of G_n. In fact, we have proved a slightly stronger result: we have proved divisibility by an extra factor of 3 when n is 0 mod 3.

Currently, I am working on proving divisibility of the number of matchings by 2 for this family of graphs, and I am close to proving divisibility by 2 when n is 0 or 1 mod 3.

Other Current Explorations

Enumerative combinatorics In addition to the above-mentioned directions, I am currently exploring the field of combinatorics of partitions, particularly combinatorial proofs of partition-related identities. Ramanujan's Lost Notebook contains several hundred q-series identities. Twenty-seven of these are relatives of the Rogers-Fine identity. Berndt and Andrews proved these 27 identities in their first volume on

Ramanujan's Lost Notebook. Berndt suggested that there should be combinatorial proofs of these identities. Ae Ja Yee (who was a post-doc under Berndt at UIUC) proved 18 of them combinatorially [9]. Recently, my colleague Chadwick Gugg (who is a graduate student in number theory at UIUC) and I have been reading and understanding Yee's combinatorial proofs and then trying our hand at the 9 identities for which combinatorial proofs still do not exist.

I am working on coming up with combinatorial proofs for a number of other questions. For example, proving any facts using combinatorial proofs for the number of spanning trees of a hypercube. I am also looking into the number of antichains in a product of four chains. The closed formulas for the number of antichains in a product of 3 or less chains are known.

This work fits well with my particular interest in enumerative combinatorics, and I hope to actively engage in this area in my post-graduate years. My interest in this area has been recently especially stirred because of enumerative questions that some of my colleagues in algebra and geometry have asked me. Some of these questions can serve as seeds for collaborative work across different fields of mathematics, which I am eager to explore.

Destroying directed cycles in a graph I was introduced to this problem by my advisor and I have been working on it. The problem is this: consider a directed graph with no directed cycles of size at most 3, and whose underlying graph is missing k edges from the complete graph. It has been conjectured by Chudnovsky and Seymour that deletion of at most $k/2$ edges of this directed graph suffices to destroy every directed cycle. They have proved the weaker statement that one never needs to delete more than k edges.

References

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