SOLUTIONS FOR QUIZ 11

Compute the number of integers solutions to the equation $x_1+x_2+x_3 = 12$, where $0 \le x_1 \le 5$, $0 \le x_2 \le 6$, and $0 \le x_3 \le 4$.

Show your work!

Hint. The number of nonnegative integer solutions to the equation $t_1 + \ldots + t_k = n$ equals $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$.

Denote by U the set of all nonnegative integer solutions to the equation $x_1 + x_2 + x_3 = 12$. For $k = 1, 2, 3$, denote by A_k the set of all the solutions with $x_1 \ge 6$, resp. $x_2 \ge 7$, $x_3 \ge 5$. We need to compute $|\overline{A_1 \cup A_2 \cup A_3}|$. By Inclusion-Exclusion Principle,

$$|\overline{A_1 \cup A_2 \cup A_3}| = |U| + \sum_{k=1}^{3} (-1)^k \sum_{i_1 < \ldots < i_k} |A_{i_1} \cap \ldots \cap A_{i_k}|. \tag{1}$$

By the hint, $|U| = \binom{12+3-1}{3-1} = \binom{14}{2}$. To compute $|A_1|$, consider $y_1 = x_1 - 6$. Any element of A_1 corresponds to a triple (y_1, x_2, x_3) of nonnegative integers, satisfying

$$y_1 + x_2 + x_3 = (x_1 - 6) + x_2 + x_3 = 12 - 6 = 6,$$

hence $|A_1| = \binom{6+3-1}{3-1} = \binom{8}{2}$. Similarly, $|A_2| = \binom{7}{2}$, and $|A_3| = \binom{9}{2}$. In the same fashion, any element of $A_1 \cap A_3$ corredponds to a nonnegative integer solution to $y_1 + x_2 + y_3 = 12 - 6 - 5 = 1$, hence $|A_1 \cap A_3| = \binom{3}{2} = 3$. The set $A_1 \cap A_2$ corresponds to the set of nonnegative integers solutions to $y_1 + y_2 + x_3 = 12 - 6 - 7 = -1$, hence this is an empty set, and $|A_1 \cap A_2| = 0$. Furthermore, $A_2 \cap A_3$ corresponds tot he nonnegative integer solutions to $x_1 + y_2 + y_3 = 12 - 7 - 5 = 0$, hence $|A_2 \cap A_3| = 1$. Finally, the set $A_1 \cap A_2 \cap A_3$ is empty, since $6 + 7 + 5 = 18 > 12$. Plugging all this into (1), we obtain:

$$|\overline{A_1 \cup A_2 \cup A_3}| = \binom{14}{2} - \binom{9}{2} - \binom{8}{2} - \binom{7}{2} + 3 + 1.$$

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