SOLUTIONS FOR HOMEWORK 11

7.6.3. Denote by U the set of all nonnegative integer solutions to the equation $x_1 + x_2 + x_3 = 13$. For $i \in \{1, 2, 3\}$, denote by A_i the set of all $(x_1, x_2, x_3) \in U$ with the property that $x_i \geq 6$. We have to compute the cardinality of the complement of $A_1 \cup A_2 \cup A_3$ in the "universe" U. By Inclusion-Exclusion Principle,

$$|\overline{A_1 \cup A_2 \cup A_3}| = |U| + \sum_{k=1}^{3} (-1)^k \sum_{i_1 < \ldots < i_k} |A_{i_1} \cap \ldots \cap A_{i_k}|. \tag{1}$$

By Section 5.5, $|U| = \binom{13+3-1}{3-1} = \binom{15}{2}$. To compute $|A_1|$, consider $y_1 = x_1 - 6$. Any element of A_1 corresponds to a triple (y_1, x_2, x_3) of nonnegative integers, satisfying

$$y_1 + x_2 + x_3 = (x_1 - 6) + x_2 + x_3 = 13 - 6 = 7,$$

hence $|A_1| = \binom{7+3-1}{3-1} = \binom{9}{2}$. Similarly, $|A_2| = |A_3| = \binom{9}{2}$. In the same fashion, any element of $A_1 \cap A_2$ corresponds to a nonnegative integer solution to $y_1 + y_2 + x_3 = 13 - 2 \cdot 6 = 1$, hence $|A_1 \cap A_2| = \binom{3}{2}$, and in the same way, $|A_1 \cap A_3| = |A_2 \cap A_3| = \binom{3}{2}$. Finally, the set $A_1 \cap A_2 \cap A_3$ is empty, since $6 + 6 + 6 = 18 > 13$. Plugging all this into (1), we obtain:

$$\left|\overline{A_1 \cup A_2 \cup A_3}\right| = \binom{15}{2} - 3 \cdot \binom{9}{2} + 3 \cdot \binom{3}{2} = 105 - 3 \cdot 36 + 3 \cdot 3 = \mathbf{6}.$$

7.6.6. Suppose an integer n is divisible by k^2, with $k > 1$ integer. Writing k as a product of primes, we conclude that n is divisible by p^2, for some prime p. Moreover, $p \leq \sqrt{n}$. Therefore, any integer less than 100 which is not squarefree is divisible by p^2, where p is a prime not exceeding $\sqrt{100} = 10$. There are four such primes: 2, 3, 5, and 7. Denote by A_1, A_2, A_3, and A_4 the sets of the integers between 1 and 99, divisible by 2^2, 3^2, 5^2, and 7^2, respectively. By the above, the set of all squarefree integers is the complement of $A_1 \cup A_2 \cup A_3 \cup A_4$ in $U = \{1, \ldots, 99\}$. The Inclusion-Exclusion Principle implies

$$|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = |U| + \sum_{k=1}^{4} (-1)^k \sum_{i_1 < \ldots < i_k} |A_{i_1} \cap \ldots \cap A_{i_k}|. \tag{2}$$

We have:

$$|A_1| = \left\lfloor \frac{99}{2^2} \right\rfloor = 24, \ |A_2| = \left\lfloor \frac{99}{3^2} \right\rfloor = 11, \ |A_3| = \left\lfloor \frac{99}{5^2} \right\rfloor = 3, \ |A_4| = \left\lfloor \frac{99}{7^2} \right\rfloor = 2,$$

$$|A_1 \cap A_2| = \left\lfloor \frac{99}{2^2 \cdot 3^2} \right\rfloor = 2, \ |A_1 \cap A_3| = \left\lfloor \frac{99}{2^2 \cdot 5^2} \right\rfloor = 0,$$

and similarly, $|A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = 0$. Similarly, one shows that the intersections of any three (or more) of our sets are empty. Thus, (2) yields $\left|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}\right| = 99 - (24 + 11 + 3 + 2) + 2 = \mathbf{61}$.

7.6.17. For $k \in \{0, 1, 2, 3, 4\}$, denote by A_k the set of all arrangements in which $2k$ remains in its proper place. We have to compute the cardinality of the complement of $A_0 \cup \ldots \cup A_4$ in the universe U of all arrangements of 10 digits. Clearly, $|U| = 10!$. If $0 \leq i_1 < \ldots < i_k \leq 4$ ($1 \leq k \leq 5$), $A_{i_1} \cap \ldots \cap A_{i_k}$ is the set of all arrangements where $2i_1, \ldots, 2i_k$ retain their positions, hence $|A_{i_1} \cap \ldots \cap A_{i_k}| = (10-k)!$. By Inclusion-Exclusion Principle,

$$\left|\overline{A_0 \cup \ldots \cup A_4}\right| = |U| + \sum_{k=1}^{5} (-1)^k \sum_{i_1 < \ldots < i_k} |A_{i_1} \cap \ldots \cap A_{i_k}|$$

$$= 10! + \sum_{k=1}^{5} \binom{5}{k} (-1)^k (10-k)! = \mathbf{10! - 5 \cdot 9! + 10 \cdot 8! - 10 \cdot 7! + 5 \cdot 6! - 5!}.$$

(here, $\binom{5}{k}$ is the number of k tuples $0 \leq i_1 < \ldots < i_k \leq 4$).

7.6.24. (*This is a bonus problem – very little partial credit is given*): We count the number of ways to arrange n integers between 1 and n in a row. Denote the set of such rearrangements by $\mathcal{S}$. Note that $|\mathcal{S}| = P(n, n) = n!$. To look at computing $|\mathcal{S}|$ from a different angle, we view a rearrangement of $\{1, \ldots, n\}$ as a function from $\{1, \ldots, n\}$ onto itself. For $k \in \{0, \ldots, n\}$, denote by $\mathcal{C}_k$ the family of subsets of $\{1, \ldots, n\}$ of cardinality k. Then $\cup_{k=0}^{n} \mathcal{C}_k = \mathcal{C}$, where $\mathcal{C} = \mathcal{P}(\{1, \ldots, n\})$ (the set of all subsets of $\{1, \ldots, n\}$). For each $A \in \mathcal{C}$, denote by $\mathcal{S}_A$ the set of rearrangements π of $\{1, \ldots, n\}$ s.t. $\pi(i) = i$ iff $i \in A$. That is, the rearrangement π leaves the elements of A in their positions, and is a derangement of $\overline{A}$. Thus,

$$|\mathcal{S}_A| = D_{|\overline{A}|}. \tag{3}$$

Note that each rearrangement of $\{1, \ldots, n\}$ belongs to one, and only one, set $\mathcal{S}_A$. In other words, $\mathcal{S} = \cup_{A \in \mathcal{C}} \mathcal{S}_A$, and the union is disjoint. Therefore,

$$|\mathcal{S}| = \sum_{A \in \mathcal{C}} |\mathcal{S}_A| = \sum_{k=0}^{n} \sum_{A \in \mathcal{C}_k} |\mathcal{S}_A|. \tag{4}$$

However, by (3), $|\mathcal{S}_A| = D_{n-k}$ for any $A \in \mathcal{C}_k$. Moreover, $|\mathcal{C}_k| = \binom{n}{k}$. Plugging $|\mathcal{S}| = P(n, n) = n!$ into (4), we obtain:

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

7.6.26. The derangements in question are those where $1, 2, 3$ occupy the last three positions, and $4, 5, 6$ occupy the first three positions. Moreover, any arrangement of $1, 2, \ldots, 6$ satisfying these conditions will be a derangement. There are $3!$ ways to arrange $1, 2, 3$ in the three allotted spaces, and $3!$ ways to arrange $4, 5, 6$. Thus, the number of the derangements satisfying our condition is $\mathbf{(3!)^2}$.

Back to: the syllabus, the main page of the course, the assignment.