SOLUTIONS FOR HOMEWORK 2

2.3.14. **(b)** *This is a bonus problem, very little partial credit is given.* $f(m,n) = m^2 - n^2$ **is not onto**. Indeed, $f(m,n) = (m-n)(m+n)$. The numbers $m-n$ and $m+n$ differ by $2n$, hence the have the same parity. If they are both odd, then $f(m,n)$ is odd. If they are both even, then $f(m,n)$ is divisible by 4 (as a product of two even numbers. Therefore, for no pair (m,n) can we have $f(m,n) = 2$.

ALTERNATIVE SOLUTION. Suppose $f(m,n) = (m-n)(m+n) = 2$. The numbers $m-n$ and $m+n$ are integer, hence they are divisors of 2. Thus, one of the four cases holds:

(1) $m-n = 1$, $m+n = 2$.
(2) $m-n = -1$, $m+n = -2$.
(3) $m-n = 2$, $m+n = 1$.
(4) $m-n = -2$, $m+n = -1$.

Thus, in each of these four cases, we obtain a system of two equations with two unknowns. In neither case can we get an integer solution.

(d) $f(m,n) = |m| - |n|$ **is onto**. We need to show that, for every $k \in \mathbb{Z}$, there exists $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ s.t. $f(m,n) = k$. For $k \geq 0$, $f(k,0) = k$. For $k < 0$, $f(0,-k) = k$.

(e) $f(m,n) = m^2 - 4$ **is not onto**. In fact, there is no (m,n) for which $f(m,n) = 3$. Indeed, if $f(m,n) = 3$, then $m^2 = 7$, and $m = \pm\sqrt{7}$ which is not an integer (in fact, $\sqrt{7}$ is irrational).

2.3.26. **(c)** $f(x) = \lceil x/5 \rceil$. Then

$$\begin{aligned} f(-1) &= \lceil -1/5 \rceil &= 0 \\ f(0) &= \lceil 0 \rceil &= 0 \\ f(2) &= \lceil 2/5 \rceil &= 1 \\ f(4) &= \lceil 4/5 \rceil &= 1 \\ f(7) &= \lceil 7/5 \rceil &= 2 \end{aligned}$$

hence $f(\{-1,0,2,4,7\}) = \{0,1,2\}$.

2.3.36. **(a)** Note first that $f(S)$ and $f(T)$ are subsets of $f(S \cup T)$, hence $f(S) \cup f(T) \subset f(S \cup T)$. To prove the opposite inclusion, consider $y \in f(S \cup T)$. Then there exists $x \in S \cup T$ s.t. $f(x) = y$. If $x \in S$, then $y \in f(S)$. If $x \in T$, then $y \in f(T)$. Thus, $y \in f(S) \cup f(T)$.

(b) $f(S \cap T)$ is a subset of both $f(S)$ and $f(T)$, hence $f(S \cap T) \subset f(S) \cap f(T)$.

2.3.37. Consider $A = \{1,2\}$, $B = \{3\}$, $S = \{1\}$, and $T = \{2\}$. Define $f(1) = f(2) = 3$. Then $S \cap T = \emptyset$, hence $f(S \cap T) = \emptyset$. On the other hand, $f(S) = f(T) = \{3\}$, hence $f(S) \cap f(T) = \{3\}$.

4.1.6. We need to show that, for any $n \in \mathbb{N}$, $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$.

Basic step: for $n = 1$, $1 \cdot 1! = 1 = 2 - 1 = 2! - 1$.

Inductive step: assuming $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$, we have to show that $\sum_{k=1}^{n+1} k \cdot k! = (n+2)! - 1$. But

$$\sum_{k=1}^{n+1} k \cdot k! = \sum_{k=1}^{n} k \cdot k! + (n+1) \cdot (n+1)! = (n+1)! - 1 + (n+1) \cdot (n+1)!$$
$$= (n+2) \cdot (n+1)! - 1 = (n+1)! - 1,$$

as desired.

4.1.21. We need to show that, for any $n \geq 5$, $2^n > n^2$.

Basic step: for $n = 5$, $2^5 = 32 > 5^2 = 25$.

Inductive step: suppose $n \geq 5$ and $2^n > n^2$. Show that $2^{n+1} > (n+1)^2$. Note that, for $n \geq 5$, $1 < 2n < n^2/2$, hence

$$(n+1)^2 = n^2 + 2n + 1 < n^2 + 2 \cdot \frac{n^2}{2} = 2n^2.$$

Therefore, by the induction hypothesis, $(n+1)^2 < 2n^2 < 2 \cdot 2^n = 2^{n+1}$, which is what we need.

4.1.36. We need to prove that, for any positive integer n, there exists $k_n \in \mathbb{N}$ s.t. $4^{n+1} + 5^{2n-1} = 21k_n$.

Basic step: for $n = 1$, $4^2 + 5^1 = 21$ ($k_1 = 1$).

Inductive step:

$$\begin{aligned}(4^{n+2} + 5^{2n+1}) - (4^{n+1} + 5^{2n-1}) &= (25-1)5^{2n-1} + (4-1)4^{n+1} \\ &= 24 \cdot 5^{2n-1} + 3(21k_n - 5^{2n-1}) \\ &= 3 \cdot 21k_n - 21 \cdot 5^{2n-1} = 21k_{n+1},\end{aligned}$$

with $k_{n+1} = 3k_n - 5^{2n-1}$.

4.1.40. Our proof proceeds by induction on n. For every n, we have to prove the statement

$P(n)$: for any sets $A_1, \ldots, A_n, B$, $(A_1 \cup \ldots \cup A_n) \cap B = (A_1 \cap B) \cup \ldots \cup (A_n \cap B)$.

The statement $P(1)$ is a tautology: $A_1 \cap B = A_1 \cap B$. This yields the basic step of our proof. Moreover, the statement $P(2)$ holds (it is one of the distributive laws):

$$(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B).$$

To handle the inductive step, we have to show that $P(n)$ implies $P(n+1)$. For the sets $A_1, \ldots, A_n, A_{n+1}$, let $A_0 = A_1 \cup \ldots \cup A_n$. By the associativity of $\cup$,

$$(A_1 \cup \ldots \cup A_n \cup A_{n+1}) \cap B = (A_0 \cup A_{n+1}) \cap B,$$

which equals $(A_0 \cap B) \cup (A_{n+1} \cap B)$ by $P(2)$. By the induction hypothesis,

$$A_0 \cap B = (A_1 \cap B) \cup \ldots \cup (A_n \cap B).$$

Therefore,

$(A_1 \cup \ldots \cup A_n \cup A_{n+1}) \cap B = (A_0 \cap B) \cup (A_{n+1} \cap B) = (A_1 \cap B) \cup \ldots \cup (A_n \cap B) \cup (A_{n+1} \cap B)$,

which establishes $P(n+1)$.

4.1.48. The formula is false for $n = 1$. Indeed, then we have $1 = (3/2)^2/2 = 9/8$, which is false. The proof fails at the basic step of induction.

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