SOLUTIONS FOR HOMEWORK 3

4.1.58. *This is a bonus problem. Very little partial credit will be given.*

We say that n lines in a plane are *in the general position* if no two lines are parallel, and no three lines have a common point.

The proof proceeds by induction. The base step ($n = 1$) is easy: 1 line divides the plane into $2 = (1^2 + 1 + 2)/2$ regions (you can check the formula for $n = 2$ in the same manner).

Next we describe the inductive step. Suppose we have n lines in the general position, which divide the plane into K regions: $A_1, \ldots, A_K$. Add one more line (call it L) in such a way that the $n + 1$ lines are still in the general position. Theses $n + 1$ lines give rise to M regions. We show that $M = K + n + 1$.

Note that $B - A$ equals the number of regions A_i intersected by L (compare the regions divided by L into two, versus those it leaves unaltered). By rotating the plane, we can assume that L coincides with the x-axis. Let's travel along L from $-\infty$ to ∞. To the left of a certain point, L lies in the region A_{i_0}. After meeting one line, it passes to A_{i_1}. After intersecting the second line, L moves into A_{i_2}. Finally, to the right of its intersection with the n-th line, L lies in a region A_n.

So, the grand total of the regions intersected by L equals $n + 1$.

We now complete the inductive step. Suppose the number of regions equals $(n^2 + n + 2)/2$ for n lines, and prove that, for $n + 1$ lines, it must be equal to $((n + 1)^2 + (n + 1) + 2)/2$. By the above, for $n + 1$ lines

$$\begin{aligned}\#\,\text{of regions} &= \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2} \\ &= \frac{(n^2 + 2n + 1) + (n + 1) + 2}{2} = \frac{(n + 1)^2 + (n + 1) + 2}{2},\end{aligned}$$

as desired.

4.1.70. **(b)** We have to show that, for any $n \geq 2$,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n - 1}{2n} < \frac{1}{\sqrt{3n + 1}}. \tag{1}$$

Once this is established, we would conclude that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n - 1}{2n} < \frac{1}{\sqrt{3n}}$$

for any positive integer n. Indeed, for $n \geq 2$ the last inequality follows from (1), while for $n = 1$, we clearly have $1/2 < 1/\sqrt{3}$.

We prove (1) by induction. The basic step consists of verifying this inequality for $n = 2$:

$$\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8} < \frac{1}{\sqrt{7}},$$

since $(8/3)^2 = 64/9 > 63/9 = 7$.

The inductive step consists of proving that, if

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}},$$

then

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3n+4}},$$

By the induction hypothesis,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3n+1}} \cdot \frac{2n+1}{2n+2}. \tag{2}$$

We need to show that

$$\frac{2n+1}{2n+2} < \frac{\sqrt{3n+1}}{\sqrt{3n+4}}.$$

Taking the square of both sides, we see that the last inequality is equivalent to:

$$1 - \frac{3}{3n+4} = \frac{3n+1}{3n+4} > \frac{(2n+1)^2}{(2n+2)^2} = 1 - \frac{4n+3}{4n^2+8n+4}. \tag{3}$$

But

$$3(4n^2+8n+4) = 12n^2+24n+12 < 12n^2+25n+12 = (3n+4)(4n+3),$$

hence

$$\frac{3}{3n+4} < \frac{4n+3}{4n^2+8n+4},$$

which establishes (3). This, in turn, implies (2).

4.2.14. The game proceeds as follows: initially, we have a pile with $n \geq 2$ stones. At each step, we pick one of the existing piles, split it into two, and multiply the numbers in the resulting piles. Proceed until we obtain n piles, with 1 stone each. Add all the pairwise products obtained during the game. Prove that, no matter how we do the splitting, we get $n(n-1)/2$ at the end (statement $P(n)$).

The statement $P(2)$ is clear: the pile with 2 stones can only be split into two piles of 1 stone each, and $1 \cdot 1 = 2(2-1)/2$. In a similar fashion, we can check $P(3)$: on the first step, we get a pile of 2 stones, and a pile of 1 stone (yielding $2 \cdot 1$), and then, we split the pile with 2 stones into two, contributing $1 \cdot 1$. At the end, we get $2 \cdot 1 + 1 \cdot 1 = 3 = 3(3-1)/2$.

The inductive step consists of showing that $P(n+1)$ is true, provided $P(2), \ldots, P(n)$ are true ($n \geq 2$). So, suppose we split a pile with $n+1$ stones into two, with k and $n+1-k$ stones, respectively. By the induction hypothesis, further splittings of these two piles will yield $k(k-1)/2$ and $(n+1-k)(n-k)/2$ (these formulae also cover the case when k or $n+1-k$ equals 1). Thus, the total sum of products involved equals

$$k(n+1-k) + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2}$$
$$= \frac{1}{2}\Big(2k(n+1) - 2k^2 + k^2 - k + n(n+1) - nk - (n+1)k + k^2\Big) = \frac{n(n+1)}{2}.$$

This establishes $P(n+1)$.

5.1.28. Denote by n_1 (n_2) the number of license plates made up of three letters, followed by three digits (respectively, four letters, followed by two digits). By the product rule, $n_1 = 26^3 \cdot 10^3$ (26 ways to pick each of the three letters, 10 ways to pick each of the three digits), and $n_2 = 26^4 \cdot 10^2$. By the sum rule, the total number of possible license plates equals $n_1 + n_2 = 26^3 \cdot 10^3 + 26^4 \cdot 10^2$.

5.1.30. **(f)** By the product rule, we have 26^6 ways to pick the remaining 6 letters.

(g) By the product rule, we have 26^4 ways to pick the remaining 4 letters.

(h) Denote by A and B the sets of the 8-letter words which start, respectively end, with BO. Then $|A| = 26^6$ by (f), and, similarly, $|B| = 26^6$. By (g), $|A \cap B| = 26^4$. We are interested in the cardinality of $A \cup B$ (words which either start or end with BO). By Inclusion-Exclusion Principle, $|A \cup B| = |A| + |B| - |A \cap B| = 2 \cdot 26^6 - 26^4$.

5.1.40. **(a)** There are 6 ways to find a place for the bride. After the place for the bride has been selected, we have 5 more spots in the row left, and 9 people. There are 9 ways to fill the first position, 8 ways to fill the second one, ..., 5 ways to pick the fifth position. So, the total number of arrangement of 6 people (including the bride) in a row equals $6 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$.

(b) There are 6 ways to pick a place for the bride, and 5 ways to pick the place for the groom. After that, we have 4 spots, and 8 candidates, left. There are 8 ways to fill the first spot, ..., 5 ways to pick the fourth one. The total number of arrangements equals $6 \cdot 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5$.

(c) **Update:** the solution below is false. More preceisley, the paragraph below counts the number of ways to place the bride, the groom, **or both** in the picture.

Denote by A (B) the set of all arrangements where the bride (resp. the groom) is present. By part (a), $|A| = |B| = 6 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$. By part (b), $|A \cap B| = 6 \cdot 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5$. By Inclusion-Exclusion Principle,

$$|A \cup B| = |A| + |B| - |A \cap B| = 2 \cdot 6 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 - 6 \cdot 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5.$$

The correct solution There are $6 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ ways to place the bride, but not the groom, in the picture (6 spots for the bride, 8 candidates for the first position, etc.). The number of ways to place the bride, but not the groom, is the same. Thus, the total number is $2 \cdot 6 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$.

Alternative solution. We use the notation from the wrong solution above. The number of arrangements we are looking for equals

$$\begin{aligned}|(A \cup B) - (A \cap B)| &= |A \cup B| - |A \cap B| = |A| + |B| - |A \cap B| - |A \cap B| \\ &= 2 \cdot 6 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 - 2 \cdot 6 \cdot 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5.\end{aligned}$$

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